Midterm – March 2016

Exercise 1. Consider an industry in which two firms compete *à la Cournot*. The inverse demand of the good is $P(q,a) = \max\{a - q, 0\}$, where q is the total output supplied by the firms and a is an uncertain parameter that takes values 8 and 4 with equal probability. Both firms produce the good at zero cost. Firm 1 observes the realized value of a before choosing its output, while firm 2 only infers it ex-post.

(a) (15 points) Describe the Bayesian game firms face, and calculate their outputs and profits in the Bayesian equilibrium.

(b) (15 points) Does the informed firm have incentives to commit ex-ante to share its information with firm 2? (Assume that firm 1 can credibly inform firm 2 about the demand parameter realized before the output decisions are made.)

Solution. (a) In the Bayesian game:

- the set of players is $N = \{1,2\}$;

- the types of player 1 are $T_1 = \{8,4\}$ and player 2 has a single type;

- A strategy for player 1 is a pair $(q_1(8), q_1(4)) \in R_+^2$ indicating its output when it observes $a = 8$ and $a = 4$, respectively, and a strategy for player 2 is a number $q_2 \in R_+$ indicating its output.

- Given a strategy profile and $a \in \{8,4\}$ the firms payoffs (profits) are $\pi_1(q_1(a), q_2, a) = P(q_1(a) + q_2, a)q_1(a)$ and $\pi_2(q_1(a), q_2, a) = P(q_1(a) + q_2, a)q_2$.

In a Bayesian equilibrium $q_1(a)$ solves

$$\max_{q_1 \geq 0} \pi_1(q_1, q_2, a)$$

for $a \in \{8,4\}$, and q_2 solves

$$\max_{q_2 \geq 0} \frac{1}{2}\pi_2(q_1(8), q_2, 8) + \frac{1}{2}\pi_2(q_1(4), q_2, 4).$$

Hence an interior equilibrium $(q_1(8), q_1(4), q_2)$ solves the system of equations

$$\begin{aligned} 8 - 2q_1(8) - q_2 &= 0 \\ 4 - 2q_1(4) - q_2 &= 0 \\ \frac{1}{2}(8 - q_1(8) - 2q_2) + \frac{1}{2}(4 - q_1(4) - 2q_2) &= 0. \end{aligned}$$

Solving the system we get $q_1(8) = 3$, $q_1(4) = 1$, and $q_2 = 2$. Equilibrium expected profits are

$$\pi_1^* = \frac{1}{2}(8 - 3 - 2)\,3 + \frac{1}{2}(4 - 1 - 2)\,1 = 5,$$

and

$$\pi_2^* = \frac{1}{2}(8-3-2)\,2 + \frac{1}{2}(4-1-2)\,2 = 4.$$

(b) If firm 1 were to share its information with firm 2, firm 2 would be able to condition its output decision on the demand realized, i.e., a strategy for firm 2 would also be a pair of real numbers $(q_2(8), q_2(4))$*, and in a Bayesian equilibrium for* $a \in \{8,4\}$

$$q_1(a) \in \arg\max_{q_1 \geq 0} \pi_1(q_1, q_2(a), a), \;\; q_2(a) \in \arg\max_{q_2 \geq 0} \pi_2(q_1(a), q_2, a).$$

Hence

$$\begin{aligned} a - 2q_1(a) - q_2(a) &= 0 \\ a - q_1(a) - 2q_2(a) &= 0 \end{aligned}$$

for $a \in \{8,4\}$*. Solving these systems of equations we get* $q_1(a) = q_2(a) = \frac{a}{3}$*. In this case profits are*

$$\tilde{\pi}_1^* = \tilde{\pi}_2^* = \frac{1}{2}(8-2(8/3))\,8/3 + \frac{1}{2}(4-2(4/3))4/3 = \frac{40}{9}.$$

Since $\tilde{\pi}_1^* < \pi_1^* = 5$ *(see part (a)), then firm 1 has no incentives to share its information with firm 2.*

Exercise 2. Consider the contract design problem of a Principal whose revenue is a random variable taking values $x_1 = 6$, $x_2 = 18$ and $x_3 = 72$ with probabilities that depend on the effort the Agent exerts, $e \in \{1, 2\}$; specifically, $p_1(1) = p_2(1) = p_3(1) = 1/3$, whereas $p_1(2) = 0$, $p_2(2) = 1/3$, and $p_3(2) = 2/3$. The Agent's reservation utility is $\underline{u} = 0$, and his cost of effort is $v(e) = e$.

(a) (10 points) Assuming that the Principal is risk-neutral and the preferences of the Agent are represented by the von Neumann-Morgenstern utility function $u(x) = \sqrt[3]{x}$, determine the optimal contract when effort is verifiable.

(b) (20 points) Under the assumptions in (a), determine the optimal contract when effort is *not* verifiable, and calculate the cost to the Principal of non-observing effort. (Assume that due to limited liability wages cannot be negative. Also note that when the lowest revenue is realized, it is revealed that the Agent exerted the lowest effort.)

(c) (10 points) Now assume that the Principal is risk averse and the Agent's utility function is $u(x) = x$. Determine the optimal contract when effort is verifiable and when it is *not* verifiable.

Solution: (a) As established in class, when the Principal is risk neutral and the Agent is risk averse the optimal contract involves paying the Agent a fixed wage that exactly compensate his from accepting the contract and making the desired effort (i.e., that satisfies the participation constraint). Thus, the contract when the effort requested is e is $\bar{w}(e)$ such that

$$u(\bar{w}(e)) = \underline{u} + v(e),$$

that is

$$\sqrt[3]{\bar{w}(e)} = e.$$

Hence $\bar{w}(1) = 1$, and $\bar{w}(2) = 8$.

The Principal's expected profits $\Pi(e)$ are

$$\Pi(e) = \mathbb{E}(X(e)) - \bar{w}(e).$$

Since

$$\mathbb{E}(X(1)) = \frac{6}{3} + \frac{18}{3} + \frac{72}{3} = 32,$$

and

$$\mathbb{E}(X(2)) = \frac{18}{3} + \frac{2}{3}(72) = 54,$$

then

$$\begin{aligned}\Pi(1) &= 32 - 1 = 31,\\ \Pi(2) &= 54 - 8 = 46.\end{aligned}$$

Hence the optimal contract is $(e^*, \bar{w}(e^*)) = (2, 8)$.

(b) When effort is not verifiable, the optimal contract must satisfy the participation and incentive constraints. Obviously, when the desired effort is $e = 1$*, a fixed wage* $\bar{w}(1) = 1$ *satisfies these constraints. Let us calculate the optimal contract when the desired effort is* $e = 2$*. Since the* $x_1 = 6$ *occurs with probability zero when effort is* $e = 2$*, it is clearly optimal to pay the lowest possible wage* $w_1 = 0$ *in this case. Then we can calculate the wages to be paid when the realized revenue is* x_2 *and* x_3 *by solving the system of equations formed by the participation and incentive constraints:*

$$\begin{aligned}\frac{1}{3}\sqrt[3]{w_2} + \frac{2}{3}\sqrt[3]{w_3} &= 2\\ \frac{1}{3}\sqrt[3]{w_2} + \frac{2}{3}\sqrt[3]{w_3} - 2 &= \frac{1}{3}\sqrt[3]{w_2} + \frac{1}{3}\sqrt[3]{w_3} - 1.\end{aligned}$$

The solution to this system is: $w_2 = 0$*, and* $w_3 = 27$*. Hence the expected wage is*

$$\mathbb{E}(W(2)) = \frac{2}{3}(27) = 18.$$

As in part (a), the Principal's profit for $e = 1$ *is* $\Pi(1) = 32 - 1 = 31$*. But now, her profits for* $e = 2$ *is* $\Pi(2) = 54 - 18 = 36$*. Hence the optimal contract is* $(e^*, \bar{w}(e^*)) = (2, (0, 0, 27))$*. The fact that effort is not verifiable reduces the Principal's profit by 10 monetary units.*

(c) As seen in class, the optimal contract involves a franchise, that is selling the business to the Agent at a price equal to the maximum the Agent is willing to pay, y*. In order to calculate this price, note that given* y*, the Agent will have to choose the effort he exerts my solving*

$$\max_{e\in\{1,2\}} \mathbb{E}u(X(e) - y).$$

And since the Agent is risk neutral,

$$Eu(X(e) - y) = E(X(e) - y) = E(X(e)) - y.$$

Thus,

$$E(X(2)) = 54 > 32 = E(X(1))$$

implies that the optimal effort is $e^* = 2$*, and the maximum price the Agent is willing to pay for the business is*

$$\mathbb{E}(X(2)) - v(2) = 54 - 2 = 52,$$

regardless of whether or not effort is verifiable. Hence $y^* = 52$.

Exercise 3. Consider an insurance market in which all individuals have the same initial wealth, $W = 51/32$ monetary units, and the same preferences, represented by the von Neumann-Morgenstern utility function $u(x) = \ln x$, where x is the individual's disposable wealth. Each individual faces the risk of loosing one monetary unit (i.e., $L = 1$). For a fraction $\lambda^H \in (0, 1)$ of the individuals the probability of suffering this loss is $p^H = 1/2$ whereas for the remaining fraction $\lambda^L = 1 - \lambda^H$ it is $p^L = 1/4$. Insurance companies have this information, but the probability with which a particular individual may suffer the loss is the individual's private information.

(a) (10 points) Calculate the policies that will be offered in the competitive equilibrium. (You will need to calculate the deductible of a certain policy; in your calculations, try the value 3/4.)

(b) (15 points) Suppose that the government puts to a referendum a law imposing to everyone mandatory full coverage insurance, and forbids insurance companies offering other policies. Calculate this policy. (Note that insurance companies would compete to supply this policy.) identify the values of $\lambda^H \in (0, 1)$ for which such a proposal would be approved by a majority of the electorate. In you arguments, assume that an individual votes in favor of the proposal only if it improves her situation relative to the competitive equilibrium. (Hint. As part of your calculations you should get a key inequality giving you a bound on λ^H. Do not get hang up on this calculation. Use $\bar{\lambda}^H = 0.2$ is an approximately bound, and proceed to conclude and interpret your results.)

Solution: (a) As seen in class, a competitive equilibrium is separating, and involves offering the policies $(I^H, D^H) = (p^H L, 0) = (1/2, 0)$, *and* (I^L, D^L) *satisfying*

$$\begin{aligned} I^L &= p^L(L - D^L) \\ u(W - p^H L) &= (1 - p^H)u(W - I^L) + p^H u(W - I^L - D^L). \end{aligned}$$

Substituting $I^L = p^L(1 - D^L)$ *into the second equation, and using the parameter values and utility function given we get*

$$\ln\left(\frac{51}{32} - \frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{51}{32} - (1 - D^L)/4\right) + \frac{1}{2}\ln\left(\frac{51}{32} - (1 - D^L)/4 - D^L\right),$$

that is

$$\left(\frac{51}{32} - \frac{1}{2}\right)^2 = \left(\frac{51}{32} - \frac{1 - D^L}{4}\right)\left(\frac{51}{32} - \frac{1 - D^L}{4} - D^L\right).$$

Solving we get $D^L = 3/4$, *and hence* $I^L = (1 - D^L)/4 = 1/16$.

(b) The mandatory full insurance policy is

$$(\bar{I}, 0) = (\bar{p}L, 0) = (\lambda p^H + (1-\lambda)\,p^L, 0) = (\frac{1+\lambda}{4}, 0)$$

Obviously, the fraction λ *of high risk individuals are better off with this policy and will therefore vote in favor. As for the low risk individuals, they will be in favor only if their welfare with this policy* $(\bar{I}, 0)$ *is greater than with the policy* $(I^L, D^L) = (1/16, 3/4)$, *that is*

$$u\left(W - \bar{I}\right) > (1 - p^L)u\left(W - I^L\right) + p^L u\left(W - I^L - D^L\right),$$

that is

$$\ln\left(\frac{51}{32} - \frac{1+\lambda}{4}\right) > \frac{3}{4}\ln\left(\frac{51}{32} - \frac{1}{16}\right) + \frac{1}{4}\ln\left(\frac{51}{32} - \frac{1}{16} - \frac{3}{4}\right),$$

which may be written as

$$\left(\frac{51}{32} - \frac{1+\lambda}{4}\right) = \left(\frac{51}{32} - \frac{1}{16}\right)^{\frac{3}{4}}\left(\frac{51}{32} - \frac{1}{16} - \frac{3}{4}\right)^{\frac{1}{4}}$$

The largest value of λ *that satisfies this inequality is*

$$\bar{\lambda} = \frac{43}{8} - \frac{1}{8}\sqrt[4]{25}49^{\frac{3}{4}} \simeq 0.2.$$

Hence the policy proposal would be approved if either high risk individuals are a majority, i.e., $\lambda > 0.5$, *or if there are minority smaller than* $\bar{\lambda} = 0, 2$, *and would not be approved if* $\lambda \in (0.2, 0.5)$.