Final Exam

(May 30, 2016)

Exercise 1. The revenue of a risk-neutral principal is a random variable $X(e)$ taking values $x_1 = 0$ and $x_2 = 4$ with probabilities that depends on the level of effort of an agent, $e \in [0, 1]$, and are given by $p_1(e) = 1 - e$ and $p_2(e) = e$, respectively. (Note that effort may take *any value* in the interval $[0, 1]$.) The agent's preferences are represented by the von Neumann-Morgenstern utility function $u(w) = \sqrt{w}$, her reservation utility is $\underline{u} = 0$, and her costs of effort is $v(e) = e^2$.

(a) (10 points) Assume that *effort is verifiable.* Determine the optimal contract, and calculate the principal's profit and the social surplus.

Solution. Since $EX(e) = 4e$ *for* $e \in [0, 1]$*, the principal's problem is*

$$\max_{(e,w)\in[0,1]\times\mathbb{R}_+} \quad 4e - w$$
$$s.t.\ \sqrt{w} \geq e^2.$$

Since the constrain is binding at a solution, this problem is equivalent to

$$\max_{e\in[0,1]} \ 4e - e^4.$$

An interior solution to this problem solves the equation

$$4 - 4e^3 = 1.$$

Hence the optimal contract is

$$(e^*, w^*) = (1, 1).$$

The principal profit is $EX(1) - 1 = 4 - 1 = 3$*, which is equals to the social surplus since the agent captures no surplus.*

(b) (10 points) Now assume that *effort not is verifiable.* Determine the optimal contract assuming that negative wages cannot be paid due to limited liability (because the agent has no income to give to the principle). Calculate the principal's profit and the lost of social surplus due to moral hazard.

Solution. Now the principal must take into account the incentives of the agent to exert effort given the wage schedule (w_1, w_2) *it offers. The agent expected utility of accepting the principal's offer is*

$$p_1(e)\sqrt{w_1} + p_2(e)\sqrt{w_2} - v(e) = (1-e)\sqrt{w_1} + e\sqrt{w_2} - e^2.$$

Hence when the agent accepts the contract he chooses the effort he exerts by solving the problem

$$\max_{e \in [0,1]} \ (1-e)\sqrt{w_1} + e\sqrt{w_2} - e^2.$$

Hence

$$\hat{e}(w) = \frac{\sqrt{w_2} - \sqrt{w_1}}{2}.$$

Note that effort decreases with $w_1 \geq 0$ *and increases with* $w_2 \geq 0$.

Since effort decreases with $w_1 \geq 0$, *it is optimal for the principal to set* $w_1 = 0$. *Denote* $w_2 = w$. *Therefore the principal must choose* w *in order to solve*

$$\max_{w \in \mathbb{R}_+} \ 4\hat{e}(0, w) - w = 2\sqrt{w} - w.$$

A solution to this problem satisfies the equation

$$\frac{1}{\sqrt{w}} = 1,$$

which implies a wage equal to 1. Hence the optimal contract is $(\hat{e}, \hat{w}) = (\hat{e}(0,1), 1) = (1/2, 1)$. *The profit to the principal is* $EX(1/2) - 1 = 2 - 1 = 1$, *and the agent captures a surplus equal to*

$$\hat{e}(0, \hat{w})\hat{w} - \hat{e}(0, \hat{w})^2 = \frac{1}{2}(1) - (1/2)^2 = 1/4.$$

(c) (10 points) Now assume the principal's and agent's preferences are described by the utility functions $\pi(x) = \ln x$, and $u(w) = w$, respectively. Identify the principal's optimal contract when effort is verifiable and when it is not.

Solution. Since the principal is risk averse and the agent is risk neutral, whether effort is verifiable or not, the optimal contract is a franchise. We just need to calculate the price the agent is will to pay for the franchise. In order to do this, we calculate the optimal effort assuming that the agent chooses it. In this case the level of effort solves the problem

$$\max_{e\in[0,1]} u(\mathbb{E}X(e)) - v(e) = 4e - e^2.$$

Since this function is strictly increasing in $\tilde{e}$ *on* $[0,1]$*, the solution to this problem is* $e^* = 1$*. Thus, the agent is willing to by the franchise for a price, which we denote as* μ*, so long as*

$$u(EX(e^*) - \mu) - v(e^*) = (4(1) - \mu) - (1)^2 \geq 0$$

Hence the agent is will to pay up to $\tilde{\mu} = 3$ *euros for the franchise. (In the notation used in the lectures, the optimal contract for the principal is* $(\tilde{e}, \tilde{w}_1, \tilde{w}_2) = (1, -3, 3)$*. Stricto senso, this contract is not feasible due to limited liability, but note that* $\tilde{w}_1 = -3$ *occurs with probability zero.)*

Exercise 2. Consider the agency problem described in the initial setting of exercise 1, except that there are agents of two types, H and L, identical to the agent described above except for their cost of effort: the cost of the H type is the function given above, but the cost of the L type is twice as much. Also assume that *effort is verifiable.*

(a) (10 points) Identify the contracts that the principal will offer to each type assuming that he *observes* the agent's type. (Use your results from part (a) of exercise 1.) Illustrate your findings providing a graph of the supply and demand schedules for agents of each type in the plane (e, w). Calculate the principals profit and the social surplus.

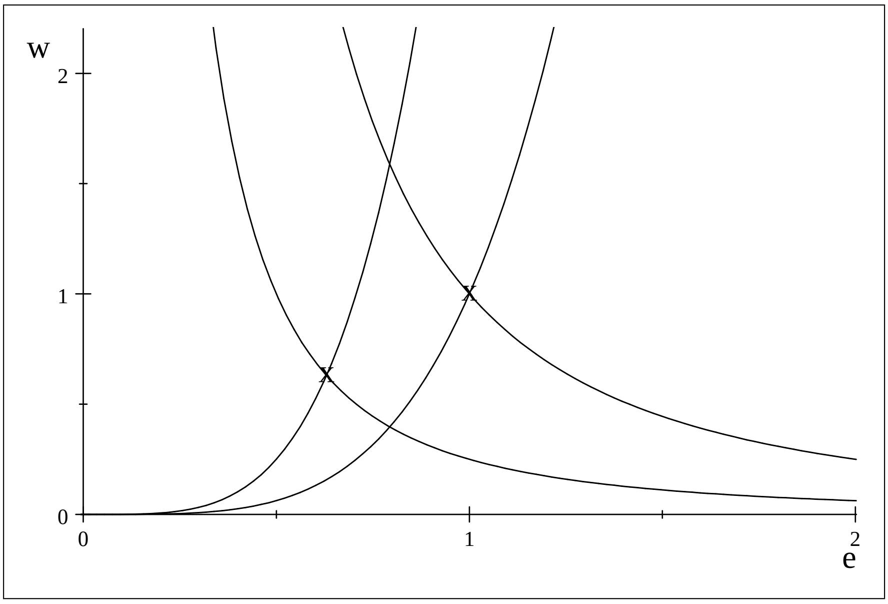
Solution. As calculated in part (a), the optimal contract for type H is $(e_H^*, w_H^*) = (1, 1)$. The optimal contract for type L solves

$$\max_{(e,w)\in[0,1]\times\mathbb{R}_+} \quad 4e - w$$
$$s.t.\ \sqrt{w} \geq 2e^2.$$

That is,

$$\max_{e\in[0,1]} \ 4e - 4e^4,$$

which solution is $e_L^* = (1/4)^{1/3}$. Hence $(e_L^*, w_L^*) = ((1/4)^{1/3}, 2(1/4)^{2/3}) \simeq (0.63, 0.79)$.



The principal profit, which equals the social surplus, is

$$S^* = \frac{1}{2}\left(4(1) - 1\right) + \frac{1}{2}\left(4(\frac{1}{4})^{1/3} - 2(\frac{1}{4})^{2/3}\right) = 2.3631.$$

(b) (20 points) Now assume that the principal *does not observe* the agent's type, and that both types are present in equal measures in the population of agents. Assuming that he wants to offer a menu of contracts to hire both types of agents, write down the principal's problem, and the system of equations identifying the optimal menu. An approximate solution to this system of equations is $[(\tilde{e}_H, \tilde{w}_H), (\tilde{e}_L, \tilde{w}_L)] = [(0.92, 1.18), (0.489, 0.229)]$. Locate the optimal screening menu in the graph, and discuss the differences of this menu and that obtained in part (c). Calculate the principal's profit and verify that this menu is superior to offering a single contract. Calculate the lost in social surplus caused by adverse selection.

Solution. The principals problem is

$$\max_{[(e_H,w_H),(e_L,w_L)]\in([0,1]\times\mathbb{R}_+)^2} \frac{1}{2}(4e_H - w_H) + \frac{1}{2}(4e_L - w_L)$$

$$\begin{aligned} s.t.& \\ \sqrt{w_L} &\geq 2e_L^2 \qquad (PC_L) \\ \sqrt{w_H} &\geq e_H^2 \qquad (PC_H) \\ \sqrt{w_L} - 2e_L^2 &\geq \sqrt{w_H} - 2e_H^2 \qquad (IC_L) \\ \sqrt{w_H} - e_H^2 &\geq \sqrt{w_L} - e_L^2 \qquad (IC_H). \end{aligned}$$

The optimal menu of contracts is identified by system of equations:

$$\begin{aligned} 4 &= \frac{2e_H}{\frac{1}{2\sqrt{w_H}}} \\ 4 &= \frac{4e_L}{\frac{1}{2\sqrt{w_L}}} + \frac{\frac{1}{2}}{1-\frac{1}{2}}(2-1)\frac{2e_L}{\frac{1}{2\sqrt{w_H}}} \\ \sqrt{w_L} &= 2e_L^2 \qquad (IC_L) \\ \sqrt{w_H} - e_H^2 &= \sqrt{w_L} - e_L^2. \qquad (IC_H) \end{aligned}$$

The principal's profit with the optimal screening menu $[(\tilde{e}_H, \tilde{w}_H), (\tilde{e}_L, \tilde{w}_L)] = [(0.92, 1.18), (0.489, 0.22$ *is*

$$\frac{1}{2}(4(0.92) - 1.18) + \frac{1}{2}(4(0.489) - 0.229) = 2.1135.$$

The principal's profit if he offers only the optimal contract acceptable by type H agents, $(e_H^*, w_H^*) = (1, 1)$, *is*

$$\frac{1}{2}(4(1) - 1) = 1.5.$$

Hence the optimal screening menu is indeed optimal.

The surplus of the agents of type H is

$$\frac{1}{2}\left(w_H - e_H^2\right) = \frac{1}{2}\left(1.18 - (0.92)^2\right) = 0.1668.$$

Hence the social surplus

$$\tilde{S} = 2.1135 + 0.1668 = 2.2803 < 2.3631 = S^*.$$

Exercise 3. In a competitive market there are 3 buyers who want to by a single unit of the good and three sellers each of whom has one unit of the good. Two of the sellers have a unit of high quality good, and the remaining seller has one unit of low quality good (L). The buyers' values for each quality of the good are $u_H = 10$ and $u_L = 1$, respectively. The cost to sellers of high quality is $c_H = 8$, and the cost to the seller of low quality is $c_L = 0$. Buyers do not observe the quality of the good at the moment of purchase – they discover it only after they buy it and use it.

(a) (10 points) Provide a graph the supply and demand schedules in this market and calculate the competitive equilibria (there may be multiple ones). Make sure to identify the price and quantity traded of each quality in each equilibrium.

Solution: The supply schedule is

$$S(p) = \begin{cases} 1 & \text{if } p \in (0,8) \\ \{1,2,3\} & \text{if } p = 8 \\ 3 & \text{if } p > 8. \end{cases}$$

The demand schedule is

$$D(p) = \begin{cases} 3 & \text{if } p \in (0,1) \\ \{0,1,2,3\} & \text{if } p = 1 \\ 0 & \text{if } p > 1. \end{cases}$$

The unique competitive equilibrium price and quantity are $p^* = q^* = 1$.

(b) (10 points) Assume that the government makes a public offer to pay two euros per unit to any seller of the good willing to supply before the market opens. Determine the impact of this program assuming that buyers observe the number of sellers present in the market at its opening. (You have to consider each seller's choice whether to sell to the government before the market opens, or to wait for the market to open and try to sell her unit in the market.) Calculate the equilibrium, and determine the social surplus and the winners and losers of this program.

Note that the government is offering to small a price for any high quality seller to supply its unit. Hence the issue is whether the low quality seller will supply its unit to the government or wait for the market to open.

If the low quality seller does not supply its units, at the market open the buyers will observe that all three sellers remain in the market, and will form their expectations as in part (a), and the resulting equilibrium would be the same. If the sellers see only two sellers present in the market, then the will infer that the seller of low quality had sold its unit to the government – the government is offering to small a price for any high quality seller to supply its unit. Therefore the supply and demand schedules at that point would be

$$\hat{S}(p) = \begin{cases} 0 & \text{if } p \in [0,8) \\ \{0,1,2\} & \text{if } p = 8 \\ 2 & \text{if } p > 8 \end{cases}, \quad \hat{D}(p) = \begin{cases} 3 & \text{if } p \in [0,10) \\ \{0,1,2,3\} & \text{if } p = 10 \\ 0 & \text{if } p > 1. \end{cases}$$

Thus, the market equilibrium would be $\hat{p}^* = 10$, $\hat{q}^* = 2$.

The seller of quality L should anticipate this outcomes, and hence infer that he either sells to the government at a price of 2 euros or sells in the market at a price of 1 euro. He will therefore choose to sell to the government.

Hence, the result of this program is that all units sell; low quality sells to the government and high quality to the buyers. The social surplus is

$$2\,(10-8) + u - 2 = 2 + u,$$

where u is the value to the government of the unit of low quality it buys. The social surplus increases to

$$2 + u - 1 \geq 0.$$

The program obviously benefits the sellers of both types. Moreover, if does not affect the buyers surplus.

Exercise 4. In a perfectly competitive labor market half of the workers are high skilled (H) and the other half are low skilled (L). A firm that hires a high (low) skilled worker obtains an expected revenue of $x_H = 12$ (respectively, $x_L = 4$) euros. The reservation utilities of high (low) skilled workers are $u_H = 6$ (respectively, $u_L = 0$). Workers may signal their skill by taking an action $y \in \mathbb{R}_+$ at cost $c_H(y) = y/4$ and $c_L(y) = y$, respectively.

(a) (10 points) Compute the *pooling* PBNE in which both types of workers are hired.

Solution. In the signaling game a strategy for the agent is a function $s : \{H, L\} \to R_+$.

In a pooling equilibrium the agents signal is $y_H = y_L = 0$. Under perfect competition a firm's expected profit is zero. Agents of type H and L accept a wage offer so long as it is greater or equal to $u_H = 6$ and $u_L = 0$, respectively. Hence in a pooling equilibrium $w \geq u_H$.

Assume that all firms offer the wage $w \in [u_H, \bar{x}]$, where

$$\bar{x} = \frac{1}{2}(12) + \frac{1}{2}(4) = 8.$$

Then all agents of both type accept the wage offer. If $w < \bar{x}$, then firms profits are positive. Hence in a perfectly competitive market $w = 8$ is a pooling equilibrium.

(b) (10 points) Compute the most efficient (*signaling)* separating PBNE. Which of the two PBNE would each type of worker prefer?

Solution. In a separating equilibrium $s(L) = 0$ and $s(H) = \bar{y} > 0$, *and firms offer the wages and beliefs are*

$$w(y) = \begin{cases} x_L & \text{if } y < \bar{y} \\ x_H & \text{if } y \geq \bar{y}, \end{cases}, \quad \mu(y) = \begin{cases} 0 & \text{if } y < \bar{y} \\ 1 & \text{if } y \geq \bar{y}, \end{cases}$$

where $\mu(y)$ is the probability that the agent is of type H. Since neither type of agent must have an incentive to imitate the other type, the inequalities

$$x_L \geq x_H - \bar{y}, \ x_H - \frac{\bar{y}}{4} \geq x_L,$$

that is

$$\bar{y} \in [x_H - x_L, 4(x_H - x_L) = [8, 32].$$

Hence in the most efficient separating equilibrium $\bar{y} = 8$.

In this equilibrium the surplus of an agent of type H is

$$x_H - \bar{y} = 12 - \frac{8}{4} = 10 > 8 = \bar{x},$$

and the surplus of an agent of type L is

$$x_L = 4 < 8 = \bar{x}.$$

Hence agents of type H are better off, while of type L are worse off, in this most efficient separating equilibrium than in the pooling equilibrium identified in part (a).