Final Exam

(May 21, 2015)

Exercise 1. A risk-neutral Principal is considering hiring an Agent to run a business project that would require an investment of 10 euros, and would yield a return of 40 euros if it succeeds, and zero euros if it fails. The success probability of the project is $p(1) = 3/5$ if the Agent exerts effort, and it is $p(0) = 1/5$ if she does not. The Agent is risk-neutral, her cost of exerting effort is 8, and her reservation utility is $\underline{u} = 0$. Effort is not verifiable.

(a) (10 points) Identify the contract specifying the agents wages when the project is successful and when it fails, (w_s, w_f), that induces the Agent to exert effort and involves the largest possible value for w_s. (Recall that incentive constraint is not binding since both the Principal and the Agent are risk neutral). Calculate the Principal's profit.

Solution: The Principal's profit is

$$\Pi\left(e, (w_s, w_f)\right) = 40p(e) - 10 - p(e)w_s - (1 - p(e))w_f$$

If the Principal makes the investment, then he chooses $[e, (w_s, w_f)] \in \{0, 1\} \times \mathbb{R}^2$ *in order to maximize* $\Pi\left(e, (w_s, w_f)\right)$ *and satisfy the participation and incentives constraints. Otherwise Principal and Agent do not contract.*

For $e = 0$ *we have*

$$\begin{aligned}
\Pi\left(0, (w_s, w_f)\right) &= 40p(0) - 10 - p(0)w_s - (1 - p(0))w_f \\
&= 40\left(\frac{1}{5}\right) - 10 - \frac{1}{5}w_s - (1 - \frac{1}{5})w_f \\
&= -2 - \frac{1}{5}w_s - (1 - \frac{1}{5})w_f.
\end{aligned}$$

Since the Agent's participation constraint requires

$$\frac{1}{5}w_s + (1 - \frac{1}{5})w_f \geq 0,$$

then

$$\Pi\left(0, (w_s, w_f)\right) \leq -2.$$

Thus, the project is viable only if it is profitable when the Agent exerts effort, i.e., $e = 1$. *In this case the wage offers* (w_s, w_f) *must satisfy the participation and incentive constraints:*

$$\begin{aligned}
\frac{3}{5}w_s + \frac{2}{5}w_f - 8 &\geq \frac{1}{5}w_s + \frac{1}{5}w_f \ (IC) \\
\frac{3}{5}w_s + \frac{2}{5}w_f - 8 &\geq 0. \ (PC)
\end{aligned}$$

As discussed in class, PC is binding, but there are multiple contracts that satisfy IC and yield identical profits. Using the participation constraint with equality we get

$$w_s = \frac{40 - 2w_f}{3}.$$

Substituting the value of w_s in the incentive constraint it becomes

$$\frac{3}{5}\left(\frac{40 - 2w_f}{3}\right) + \frac{2}{5}w_f - 8 \geq \frac{1}{5}\left(\frac{40 - 2w_f}{3}\right) + \frac{4}{5}w_f.$$

Simplifying this inequality yields

$$w_f \leq -4.$$

Hence the contract we are looking for is $[e, (w_s, w_f)] = [1, (-4, 16)]$. With this contract the Principal's profit is

$$\Pi\left(1, (-4, 16)\right) = 40\left(\frac{3}{5}\right) - 10 - \left(\frac{3}{5}\right)16 + \left(\frac{2}{5}\right)4 = 6.$$

Therefore the Principal will make the investment and hire the Agent.

(b) (10 points) Now assume that the Agent has "limited liability"; that is, she has a collateral of $C \geq 0$ euros, and therefore her wage must satisfy $w > -C$ whether the business project succeeds or fails. Identify the contract that would induce the Agent to exert effort, and determine the values values of C for which the Principal will make the investment and hire the Agent.

If $C < 4$, then the contract identified in part (a) is not feasible – the Agent has not enough money to pay the Principal ($w_f = -4$) if the project fails. Since $C \geq 0$, the smallest wage w_f feasible is $w_f = -C$. Thus, w_s must satisfy the inequalities

$$\begin{aligned}
\frac{3}{5}w_s - \frac{2}{5}C - 8 &\geq \frac{1}{5}w_s - \frac{4}{5}C \\
\frac{3}{5}w_s - \frac{2}{5}C - 8 &\geq 0
\end{aligned}$$

that is,

$$w_s = \max\left\{\frac{5}{2}\left(8 - \frac{2}{5}C\right), \frac{5}{3}\left(8 - \frac{2}{5}C\right)\right\} = \frac{5}{2}\left(8 - \frac{2}{5}C\right) = 20 - C.$$

(Note that the participation constraint is not binding.) Profits are

$$\Pi\left(1, (-C, 20 - C)\right) = 40\left(\frac{3}{5}\right) - 10 - \left(\frac{3}{5}\right)(20 - C) - \left(\frac{2}{5}\right)(-C) = C + 2 \geq 2.$$

Hence for all $C \geq 0$ the profits are positive, and therefore the Principal will make the investment and hire the Agent.

Exercise 2. Consider the contract design problem of a risk-neutral Principal who wants to hire an Agent. There are two types of agents, H and L, both with the same preferences, which are represented by the Bernuilli utility function $u(x) = x$, and the same reservation utilities, given by $\underline{u} = 1$, but with different costs of effort, $v_i(e) = k_i e$ with $k_H = 1$ and $k_L = 2$. Let q denote the probability that the Agent is of type H. The Principal's expected revenue is a function of the Agent's effort, which is verifiable, and is given by $\bar{x}(e) = 4 \ln e$ for $e \in [1, \infty)$.

(a) (15 points) Calculate the contract the Principal will offer to each type of agent if types were observable. Illustrate your findings providing a graph of the Principal's demand of effort for each type of agent as well as the supply of effort of each type of agent.

Solution: With complete information the optimal contracts $[(e_H^*, w_H^*), (e_L^*, w_L^*)]$ *solve the systems*

$$\begin{aligned} \frac{4}{e_H} &= \frac{1}{1} \\ w_H &= e_H + 1 \ (PC_H) \end{aligned}$$

and

$$\begin{aligned} \frac{4}{e_L} &= \frac{2}{1} \\ w_L &= 2e_L + 1 \ (PC_L). \end{aligned}$$

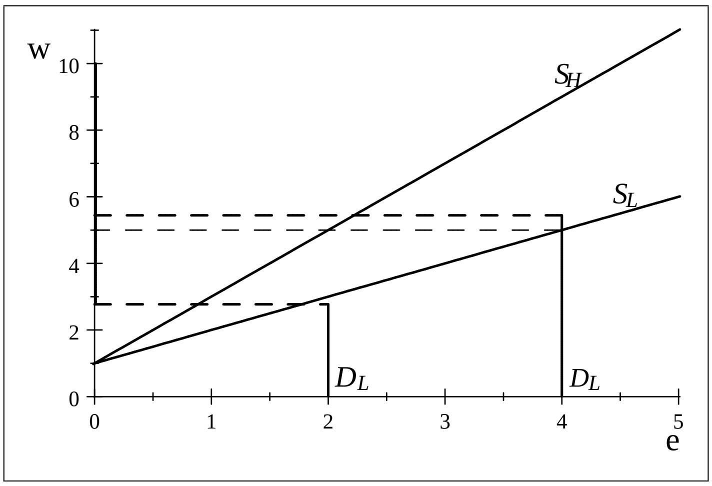
Hence $[(e_H^*, w_H^*), (e_L^*, w_L^*)] = [(4, 5), (2, 5)]$. *However, while*

$$\pi(e_H^*, w_H^*) = 4 \ln 4 - 5 > 0,$$

we have

$$\pi(e_L^*, w_L^*) = 4 \ln 2 - 5 < 0.$$

Therefore the Principal will not hire Agent L.



(b) (15 points) Identify the optimal menu of contracts the Principal will offer if Agents' types are private information (that is, not observed by the Principal), assuming that the Principal wants to hire the Agent whichever may be her type.

Solution: As seen in class, the system of equations identifying the optimal menu with asymmetric information is:

$$\begin{aligned} \frac{4}{e_H} &= 1 \\ \frac{4}{e_L} &= 2 + \frac{q}{1-q} \\ w_L &= 2e_L + 1 \\ w_H - e_H &= w_L - e_L \end{aligned}$$

Solving the system we get

$$[(\tilde{e}_H, \tilde{w}_H), (\tilde{e}_L, \tilde{w}_L)] = \left[\left(4, \frac{14-9q}{2-q}\right), \left(\frac{4(1-q)}{2-q}, \frac{10-9q}{2-q}\right)\right].$$

(c) (10 points) Determine for which values of q the menu calculated in part (b) yields more profits than hiring only the high type worker.

As shown in part (a) it is never optimal to hire Agent L. Hence the menu of part (b) is never optimal, and the optimal contract to offer is the single contract $(e_H^*, w_H^*) = (4, 5)$. *(Of course, this contract is not accepted by Agent L.)*

Exercise 3. A competitive market provides insurance to a population of individuals with preferences represented by the Bernuilli utility function $u(x) = \ln x$, where x is the individual's disposable income, who have an initial wealth $W = 5$ and face the risk of a monetary lose $L = 4$. For a fraction $\lambda \in (0,1)$ of the individuals the probability of losing L is $p^L = 1/4$ whereas for the remaining fraction $1-\lambda$ this probability is $p^H = 1/2$. This information is common knowledge to all market participants. At the time an individual is signing a policy, an insurance company does not know whether her probability of loosing L is p^L or p^H.

(a) (10 points) Which policies will be offer in a competitive equilibrium? (Assume that a competitive equilibrium exist, and identify the policies offered.) (Show your solution graphically first, and then do the algebra.)

Solution. As seen in class a competitive equilibrium, when it exists, is separating. In a separating equilibrium high risk agents get full insurance, that is $(I_H^*, D_H^*) = (p^H L, 0)$. *Hence the expected utility for a high risk individual is*

$$u(W - p^H L) = \ln\left(W - p^H L\right) = \ln\left(5 - \left(\frac{1}{2}\right)4\right) = \ln 3$$

Low risk individuals get partial insurance: $(I_L^*, D_L^*) = (p^L(L-D), D)$, *where* D *must leave the high risk individuals indifferent between the policy* (I_H^*, D_H^*) *and the policy* $(p^L(L-D), D)$, *that is,*

$$p^H u(W - p^L(L-D) - D) + \left(1 - p^H\right) u(W - p^L(L-D)) = u(W - p^H L)$$

Substituting, this equation becomes

$$\frac{1}{2}\ln\left(5 - \frac{1}{4}(4-D) - D\right) + \frac{1}{2}\ln(5 - \frac{3}{4} + \frac{D}{2}) = \ln 3.$$

which may be written as

$$\left(5 - \frac{1}{4}(4-D) - D\right)\left(5 - \frac{3}{4} + \frac{D}{2}\right) = 3^2$$

or

$$-\frac{3}{8}D^2 - \frac{19}{16}D + 8 = 0.$$

Hence

$$D^* = -\frac{4}{3}\left(\frac{19}{16} - \sqrt{\left(\frac{19}{16}\right)^2 + 12}\right) \simeq \frac{10}{3}.$$

For these policies to be a separating equilibrium, the best pooling policy, $(\bar{I}, 0) = (\bar{p}L, 0)$, *where*

$$\begin{aligned}\bar{p} &= \lambda p_L + (1-\lambda)\, p^H \\ &= \frac{\lambda}{4} + \frac{1-\lambda}{2} \\ &= \frac{1}{2} - \frac{1}{4}\lambda,\end{aligned}$$

must not be preferred by a low risk individual.

The expected utility of the low risk type with the policy $(p^L(L-D^*),D^*)$ *is*

$$\begin{aligned} p^L u(W-p^L(L-D^*)-D^*)+\left(1-p^L\right)u(W-p^L(L-D^*)) &= \frac{1}{4}\ln\left(5-\frac{1}{4}(4-\frac{10}{3})-\frac{10}{3}\right) \\ &\quad +(1-\frac{1}{4})\ln\left(5-\frac{1}{4}(4-\frac{10}{3})\right) \\ &= u_L^* \simeq 1.283. \end{aligned}$$

Hence we must have

$$u(W-\bar{p}L)\leq u_L^*,$$

that is,

$$\ln\left(5-\left(\frac{1}{2}-\frac{1}{4}\lambda\right)4\right)=\ln(3+\lambda)\leq u_L^*,$$

which may be written as

$$\lambda\leq e^{u_L^*}-3\simeq 0.6.$$

(b) (10 points) If it is mandatory that policies offer full insurance, which policies will companies offer for each value of λ? Which individuals would subscribe them? (Again, show your solution graphically first, and then do the algebra.)

If full insurance is mandatory, then companies will offer the pooling policy $(\bar{I},0)=(\bar{p}L,0)$ *if it is acceptable by the low risk individuals; that is, if*

$$u(W-\bar{p}L)\geq p^L u\left(W-L\right)+(1-p^L)u\left(W\right),$$

holds. Substituting, this inequality becomes

$$\ln(3+\lambda)\geq\frac{1}{4}\ln 1+\frac{3}{4}\ln 5=\frac{3}{4}\ln 5$$

or

$$\lambda\geq 5^{\frac{3}{4}}-3\simeq 0.34.$$

If $\lambda<5^{\frac{3}{4}}-3$, *then the insurance companies will offer the policy*

$$(I_H^*,D_H^*)=(p^H L,0)=(2,0)$$

and only high risk type individuals we get insurance.

Exercise 4. (20 points) In a competitive labor there are workers of two types, H and L. The expected revenue of a firm that hires a worker of type $i \in \{H, L\}$ with a level of education $y \in \mathbb{R}_+$ is $\bar{x}(y, i) = a_i + \sqrt{y}$, where $a_H = 2$ and $a_L = 1$. Workers choose their levels of education before entering the labor market. Firms observe the workers' level of education, but not their types, and make a wage offer. The workers payoffs are $u_i(y, w) = w - c_i(y) - \underline{u}_i$, where c_i is the worker's cost of education, which is given by $c_H(y) = y/2$ and $c_L(y) = y$, and $\underline{u}_i$ is the worker's reservation utility, given by $\underline{u}_H = \underline{u}_L = 0$. Firms payoffs are their expected profits. (Of course, in a competitive equilibrium firms' profits are zero.)

(a) (10 points) Compute the most efficient separating PBNE (that is, the PBNE in which an worker's choice of y signals her type). (Show your solution graphically first, and then do the algebra.)

Solution.

In a competitive equilibrium $w_i = \bar{x}(y_i, i)$ *for* $i \in \{H, L\}$. *Note that* $y_L^* = 1/4$ *maximizes* $\bar{x}(y, L)$, *and* $y_H^* = 1$ *maximizes* $\bar{x}(y, H)$. *In the most efficient separating PBNE* $y_L = y_L^*$ *and* $y_H := \bar{y}$ *is the smallest value in* $[0, \infty)$ *such that an* L *worker prefers to signal* y_L^* *than* y_H*; that is,*

$$w_L^* - c_L(y_L^*) \geq w_H^* - c_L(\bar{y}).$$

Substituting yields the inequality

$$(1 + \frac{1}{2}) - \frac{1}{4} \geq (2 + \sqrt{\bar{y}}) - \bar{y},$$

that is,

$$-\frac{3}{4} \geq \sqrt{\bar{y}} - \bar{y}.$$

Hence

$$\bar{y} = \frac{9}{4},$$

and

$$w_H = 2 + \sqrt{\frac{9}{4}} = \frac{7}{2}.$$

The equilibrium payoffs are

$$u_L^* = w_L^* - c_L(y_L^*) = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$$

and

$$u_H^* = w_H^* - c_H(\bar{y}) = \frac{7}{2} - \frac{9}{8} = \frac{19}{8}.$$

(b) (10 points) Assume that the fraction of workers of type L is $q \in (0,1)$. Compute the pooling PBNE in which the workers level of education $\hat{y}$ maximizes the surplus,

$$W(y) = q(\bar{x}(y,L) - c_L(y)) + (1-q)\,(\bar{x}(y,H) - c_H(y)),$$

and identify the values of q for which this equilibrium Pareto dominates the separating equilibrium identified in (a).

Solution. We have

$$\begin{aligned} W(y) &= q(1+\sqrt{y}-y) + (1-q)\,(2+\sqrt{y}-\frac{y}{2}) \\ &= 2 - q + \sqrt{y} - \frac{1+q}{2}y, \end{aligned}$$

Hence

$$W'(y) = \frac{1}{2\sqrt{y}} - \frac{1+q}{2} = 0$$

yields

$$\hat{y}(q) = \frac{1}{(1+q)^2}.$$

The pooling wage is

$$\begin{aligned} \hat{w}(q) &= q\left(1+\frac{1}{1+q}\right) + (1-q)\left(2+\frac{1}{1+q}\right) \\ &= 2 + \frac{1}{1+q} - q. \end{aligned}$$

Therefore

$$\hat{u}_L(q) = 2 + \frac{1}{1+q} - q - \frac{1}{(1+q)^2}.$$

Obviously the low type is better of in this pooling equilibrium. (This is easy to show by noting that $d\hat{u}_L(q)/dq < 0$ on $(0,1)$ and $\hat{u}_L(1) = 5/4$.)

As for the high type, we have

$$\hat{u}_H(q) = 2 + \frac{1}{1+q} - q - \frac{1}{2\,(1+q)^2} = \frac{19}{8}.$$

Therefore a high type worker is better off in the pooling equilibrium when

$$\hat{u}_H(q) > \frac{19}{8},$$

which requires

$$q \gtrapprox 0.12.$$