Final Exam

(June 1, 2017)

Exercise 1. The revenue of a risk-neutral principal is a random variable $X(e)$ taking values $x_1 = 2$ and $x_2 = 10$ with probabilities that depends on the level of effort of an agent, $e \in [0, 1]$, and are given by $p_1(e) = 1 - \sqrt{e}/2$ and $p_2(e) = \sqrt{e}/2$, respectively. There are two types of agents L and H with identical preferences represented by the von Neumann-Morgenstern utility function $u(w) = \sqrt{w}$, and identical reservation utility $\underline{u} = 0$, but different costs of effort given by $v_L(e) = e$ and $v_H(e) = 2e$.

(a) (20 points) Assume that *effort is verifiable* and the Principal observes the agent's type. Determine the contracts the principal will offer to each type of agent. Illustrate your results providing a graph the effort supply and effort demand functions for each type of agent.

Solution. For $e \in [0, 1]$,

$$E[X(e)] = 2\left(1 - \frac{\sqrt{e}}{2}\right) + 10\left(\frac{\sqrt{e}}{2}\right) = 2 + 4\sqrt{e}.$$

For $\tau \in \{H, L\}$, *the principal's problem is*

$$\max_{(e,w)\in[0,1]\times\mathbb{R}_+} \quad 2 + 4\sqrt{e} - w$$
$$s.t. \ \sqrt{w} \geq K^\tau e,$$

where $K^H = 2$, *and* $K^L = 1$. *The first order conditions for an interior solution are described by the system of equations*

$$\begin{aligned} \frac{4}{2\sqrt{e}} &= 2K^\tau\sqrt{w} \\ \sqrt{w} &= K^\tau e. \end{aligned}$$

The first equation defines the Principal's demand of effort, and the second equation defines the Agent's supply of effort.

For $\tau = L$ *these functions are*

$$\begin{aligned} w &= \frac{1}{e} \\ w &= e^2. \end{aligned}$$

Hence the optimal contract is

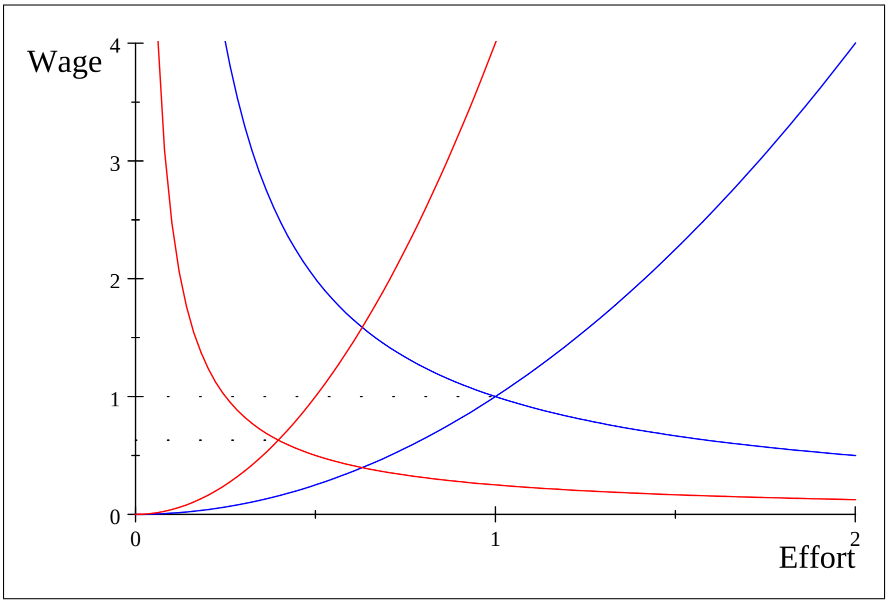
$$(e^L, w^L) = (1, 1).$$

For $\tau = H$ *these functions are*

$$\begin{aligned} w &= \frac{1}{4e} \\ w &= 4e^2. \end{aligned}$$

Hence the optimal contract is

$$(e^H, w^H) = (\frac{1}{16^{\frac{1}{3}}}, \frac{4}{16^{\frac{2}{3}}}).$$



(b) (20 points) Now assume that the Agent's type is observable, but *effort is not verifiable.* Also assume that only two efforts levels are feasible, $e = 1/4$ and $e = 1$. Determine the contracts the Principal will offer to each type of agent.

Solution. Since upon accepting a contract the lowest effort an agent can exert is $e = 1/4$, the optimal contracts for inducing agents to exert low effort, $e = 1/4$, involve fixed wages satisfying the participations constrains

$$\sqrt{w^{\tau}} = K^{\tau}\left(\frac{1}{4}\right);$$

that is, $\bar{w}^L = 1/16$, and $\bar{w}^H = 1/4$. The resulting profit for $\tau \in \{H, L\}$ are

$$E\left[X(\frac{1}{4})\right] - w^{\tau} = 2 + 4\sqrt{\frac{1}{4}} - \bar{w}^{\tau} = 4 - \bar{w}^{\tau} > 0.$$

The incentive compatible contract for the agents of type L to exert high effort ($e = 1$) solves the system $\left(\frac{1}{2}\right)x - 1 = \frac{1}{4}x - \frac{1}{4}$, Solution is: 3

$$\begin{aligned}\left(1-\frac{1}{2}\right)\sqrt{w_1} + \left(\frac{1}{2}\right)\sqrt{w_2} &= 1\\ \left(1-\frac{1}{2}\right)\sqrt{w_1} + \left(\frac{1}{2}\right)\sqrt{w_2} - 1 &= \frac{3}{4}\sqrt{w_1} + \frac{1}{4}\sqrt{w_2} - \frac{1}{4}.\end{aligned}$$

The solution to this system involves a negative wage w_1. Assuming that negative wages cannot be paid due to limited liability, forces the Principal to set up $w_1 = 0$, and hence the incentive compatibility constraint implies $w_2 = 3$. For this wage contract, $(w_1^L, w_2^L) = (0, 3)$, the Principal's profit is

$$E\left[X(1)\right] - \frac{1}{2}(3) = 6 - \frac{3}{2} = 4.5 > 4 - w^L = \frac{63}{16}.$$

Hence the optimal to offer the agents of type L is $(\tilde{e}^L, \tilde{w}_1^L, \tilde{w}_2^L) = (1, 0, 3)$.

The incentive compatible contract for the agents of type H to exert high effort solves the system

$$\begin{aligned}\left(1-\frac{1}{2}\right)\sqrt{w_1} + \left(\frac{1}{2}\right)\sqrt{w_2} &= 2\\ \left(1-\frac{1}{2}\right)\sqrt{w_1} + \left(\frac{1}{2}\right)\sqrt{w_2} - 2 &= \frac{3}{4}\sqrt{w_1} + \frac{1}{4}\sqrt{w_2} - \frac{2}{4},\end{aligned}$$

Again the solution to this system involves a negative wage w_1. Setting $w_1 = 0$, requires $w_2 = 6$ in order to satisfy the incentive compatibility constraint. The Principal's profit with this wage contract $(w_1^H, w_2^H) = (0, 6)$ is

$$E\left[X(1)\right] - \frac{1}{2}(6) = 6 - 3 < 4 - \bar{w}^H = \frac{15}{4}.$$

Hence the optimal contract to offer the agents of type H is $(\tilde{e}^H, \tilde{w}_1^H, \tilde{w}_2^H) = (1/4, 1/4, 1/4)$.

(c) (25 points) Now assume that effort is verifiable, and that only two efforts levels, $e = 1/4$ and $e = 1$, are feasible. However, the Principal *does not observe the agent's type.* Agents of type H and L are present in the population of agents in fractions $q \in (0, 1)$ and $1-q$, respectively. Identify the Principal's optimal menu of contracts for each value of q. (Keep on mind that the Principal may choose to offer a single contract, which may be acceptable either both types or only by the low cost type, if either of these contracts generates more profit than the optimal menu of contracts satisfying participation and incentive constraints.)

Solution. The Principal may offer a single "pooling" contract, which can be either the contract $(e, w) = (1/4, 1/4)$, *which both agents accept, leading to an expected profit of*

$$\Pi_H = E\left[X(\frac{1}{4})\right] - \frac{1}{4} = 4 - \frac{1}{4} = \frac{63}{16},$$

or the contract $(1, 1)$, *which only the agents of type L accept, leading to the an expected profit of*

$$\Pi_L = (1-q)\,(E[X(1)] - 1) = 5\,(1-q)\,.$$

The Principal may also design an incentive compatible menu of contracts involving low effort for the high type, $e^H = 1/4$, *and high effort for the low type,* $e^L = 1$. *As shown in class, such menu involve wages* w^L *and* w^H *that are identified by participation constraint of the type* H *and the incentive of the type* L,

$$\begin{aligned} \sqrt{w^H} &\geq 2e^H \qquad (PC_H) \\ \sqrt{w^L} - e^L &\geq \sqrt{w^H} - e^H \qquad (IC_L). \end{aligned}$$

That is,

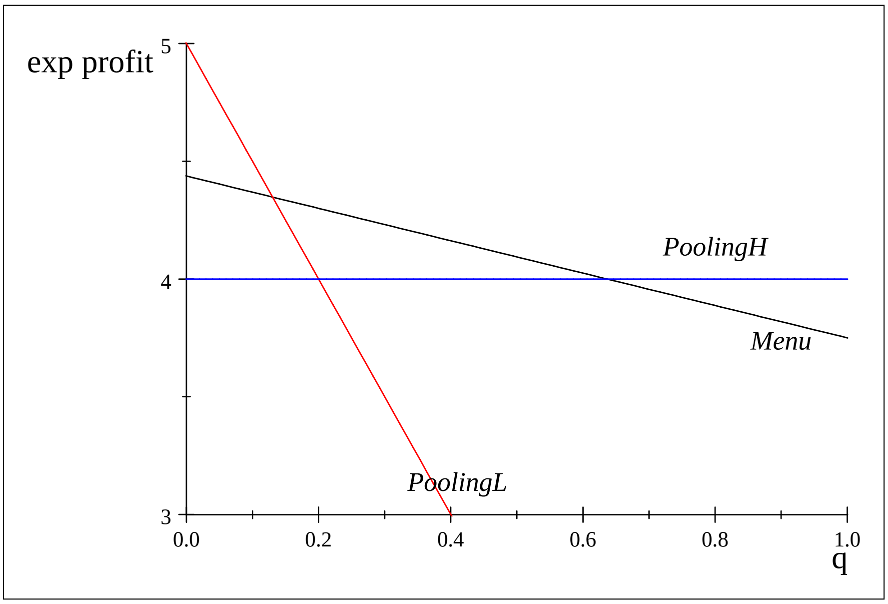
$$\begin{aligned} \sqrt{w^H} &= 2\left(\frac{1}{4}\right) \\ \sqrt{w^L} - 1 &= \sqrt{\tilde{w}^H} - \frac{1}{4}. \end{aligned}$$

whose solution is $w^H = 1/4$ *and*

$$w^L = \left(1 + \left(\sqrt{\frac{1}{4}} - \frac{1}{4}\right)\right)^2 = \frac{25}{16}.$$

For this menu of contracts, $\{(1/4, 1/4), (1, 25/16)\}$, *the expected profit is*

$$\begin{aligned} \Pi_S &= q\left(E[X(\frac{1}{4})] - \frac{1}{4}\right) + (1-q)\left(E[X(1)] - \frac{25}{16}\right) \\ &= q\left(4 - \frac{1}{4}\right) + (1-q)\left(6 - \frac{25}{16}\right) \\ &= \frac{71}{16} - \frac{11}{16}q. \end{aligned}$$



Thus, for low values of q it is optimal to offer the contract $(1,1)$. *Specifically, for q such that*

$$5(1-q) > \frac{71}{16} - \frac{11}{16}q \Leftrightarrow q < \frac{3}{23}.$$

For high values of q it is optimal to offer the contract $(1/4, 1/4)$. *Specifically, for q such that*

$$\frac{71}{16} - \frac{11}{16}q < \frac{64}{16} \Leftrightarrow q > \frac{7}{11}.$$

For intermediate values of q, that is, for $q \in (\frac{3}{23}.\frac{7}{11})$, offering the menu $\{(1/4, 1/4), (1, 25/16)\}$ *is optimal.*

Exercise 2. In any given day, tourists traveling to certain city known to be a pickpocket's playground face the risk of loosing the 32 euros they typically carry in their wallet. For the more alert tourists, this happens with probability $p_L = 1/4$, while for the inattentive ones this probability is $p_H = 1/2$. Each tourist has an allowance of $W = 100$ euros for the day, and his preferences are described by the von Neumann-Morgenstern utility function $u(x) = \ln x$.

(a) (5 points) Assume that there is a competitive insurance market where tourist may subscribe a policy covering this risk. Determine the policies that will be offered assuming that insurance companies can tell whether a tourist is of the alert or the inattentive type.

Solution: Since the market is competitive, under complete information companies will offer full insurance to each type, at a premium equal to the expected cost of the policy. These full insurance fair policies are

$$(I_H, 0) = (32p_H, 0) = (16, 0),$$

and

$$(I_L, 0) = (32p_L, 0) = (8, 0).$$

(b) (10 points) Assume that there is a competitive insurance market where tourist may subscribe a policy covering this risk. If half of the tourist are alert and the other half are inattentive and companies cannot distinguish amongst tourists of either type, which insurance policies will be offered?

Solution. As established in class, a competitive equilibrium, when it exists, offers separating fair policies $(I_H, 0) = (16, 0)$ *and* $(\hat{I}_L, \hat{D}_L)$ *such that*

$$\hat{I}_L = (32 - \hat{D}_L)p_L$$

and the inattentive tourist will be indifferent between the two policies, i.e.,

$$\frac{1}{2}\ln\left(100 - (32 - \hat{D}_L)p_L - \hat{D}_L\right) + \frac{1}{2}\ln\left(100 - (32 - \hat{D}_L)p_L\right) = \ln(100 - 16).$$

This equation may be write for $x = D_L$ *as*

$$(100 - (32 - x)/4 - x)(100 - (32 - x)/4) = (100 - 16)^2,$$

that is

$$-\frac{3}{16}x^2 - 46x + 1408 = 0.$$

Solving this equation we get $\hat{D}_L = \frac{16}{3}\sqrt{793} - \frac{368}{3} \simeq 27.521$.

For these policies to form a competitive equilibrium the alert tourist must prefer the policy $(\hat{I}_L, \hat{D}_L)$ *to the pooling policy* $(\bar{I}, 0) = (32\bar{p}, 0)$, *where*

$$\bar{p} = \frac{1}{2}p_H + \frac{1}{2}p_L = \frac{3}{8}.$$

That is $(\bar{I}, 0) = (12, 0)$.

Since the expected utility of an alert tourist with the separating policy is

$$\begin{aligned}
&\frac{1}{4}\ln\left(100-\left(32-\left(\frac{16}{3}\sqrt{793}-\frac{368}{3}\right)\right)/4-\left(\frac{16}{3}\sqrt{793}-\frac{368}{3}\right)\right)\\
&+\frac{3}{4}\ln\left(100-\left(\frac{16}{3}\sqrt{793}-\frac{368}{3}\right)/4\right)\\
\simeq\ &4.467,
\end{aligned}$$

and his expected utility with the pooling policy is

$$\ln(100-12)\simeq 4.477,$$

there is no competitive equilibrium in this market.

(c) (10 points) Assume that the market is monopolized by a single company, and that half of the tourist are alert and the other half are inattentive . Which policy will offer this company if by law cannot discriminate (i.e., must offer a single policy)? (Hint. Should the monopoly offer full insurance? Should the monopoly offer a policy intended for both types of tourists, or a policy that only inattentive tourist would subscribe?)

Solution. The company must decide with to offer a policy that only inattentive tourist subscribe or one which both types of tourists subscribe Obviously, in either case the company will offer full insurance since it can extract more surplus from the risk averse tourists. The largest premium the inattentive tourists are will to pay solves the equation

$$\ln(100-x)=\frac{1}{2}\ln(100-32)+\frac{1}{2}\ln(100),$$

that is,

$$(100-x)^2-(100-32)(100)=x^2-200x+3200=0.$$

Solving this equation we get $I_L^*=100-20\sqrt{17}\simeq 17.538$, *and expected profits are*

$$\frac{1}{2}\left(100-20\sqrt{17}-32p_H\right)=\frac{1}{2}\left(100-20\sqrt{17}-16\right)\simeq 0.76894.$$

If the firm offers a policy that both types subscribe, it has to offer it at the maximum premium the alert tourist are willing to pay, that is,

$$\ln(100-x)=\frac{1}{4}\ln(100-32)+\frac{3}{4}\ln(100),$$

or

$$(100-x)^4-(100-32)(100)^3=(100-x)^4-68000000=0,$$

that is

$$\bar{I}^*=100-\sqrt[4]{68000000}\simeq 9.1913$$

The expected profits offering this policy are

$$\bar{I}^*-32\bar{p}=\bar{I}^*-12<0.$$

Hence the company will offer the policy $(I_L^*,0)$, *which will be subscribed only by inattentive tourists.*

Exercise 3. (10 points) Calculate the seller's expected revenue, the bidders' expected payoff and the expected surplus generated in a second-price sealed-bid action of a single item in which the seller's value of the item is zero, and in which there are 2 bidders with values for the item that are independently and uniformly distributed on the interval $[0, 2]$. (Hint. The density of the bidders values X_i is $f(x) = 1/2$, and the cumulative distribution is $F(x) = x/2$.)

Solution. In this auction bidders bid their true value. Thus, the surplus generated is the largest of the two values, $Y_1^{(2)} = \max\{X_1, X_2\}$, *and the price at which the item is sold (which is the seller's revenue) is the second largest of the two values,* $Y_2^{(2)} = \min\{X_1, X_2\}$. *The cumulative distribution of* $Y_1^{(2)}$ *is,*

$$\begin{aligned} G_1^{(2)}(y) &= \Pr(X_1 \leq y, X_2 \leq y) \\ &= \Pr(X_1 \leq y) \Pr(X_2 \leq y) \\ &= F(y)^2 \\ &= \frac{y^2}{4}. \end{aligned}$$

Hence the expected revenue is

$$E[Y_1^{(2)}] = \int_0^2 y dG_1^{(2)}(y) = \int_0^2 y \left(\frac{y}{2}\right) dy = \frac{4}{3}.$$

The cumulative distribution of $Y_2^{(2)}$ *is,*

$$\begin{aligned} G_2^{(2)}(y) &= \Pr(X_1 \leq y, X_2 \leq y) + \Pr(X_1 \leq y, X_2 > y) + \Pr(X_1 > y, X_2 \leq y) \\ &= F(y)^2 + 2F(y)\left(1 - F(y)\right) \\ &= 2F(y) - F(y)^2 \\ &= y - \frac{y^2}{4}. \end{aligned}$$

Therefore the expected revenue to the seller is

$$E[Y_2^{(2)}] = \int_0^2 y dG_2^{(2)}(y) = \int_0^2 y \left(1 - \frac{y}{2}\right) dy = \frac{2}{3}.$$

The expected payoffs to the bidder can be calculated as the difference between the expected surplus and the expected revenue, divided by the number of bidders

$$\textit{Expected Payoff to a Bidder} = \frac{E[Y_1^{(2)}] - E[Y_2^{(2)}]}{2} = \frac{1}{3}.$$