Theory of the Firm

The Firm’s Problem:
Costs and Profits
Firm’s Problem: Description

• We consider a firm producing a single good Q using two inputs: L (labour) and K (capital).

• The technology of the firm is described by the production function, $F(L,K)$, which provides the maximum level of output that can be obtained for each input combination, (L,K).

• Let $p$ be the market price of good Q, $w$ the price of the labour input (L) and $r$ the price of the capital input (K).

• We assume that the firm is competitive in the input markets; i.e., (w,r) are given.
Firm’s Problem: Description

• As the goal of the firm is profit maximization (profit = revenue – cost), we can write the firm’s problem in the following way:

\[
\begin{align*}
\text{Max } pQ - wL - rK \\
\text{s.t. } & Q \leq F(L, K) \\
& Q \geq 0, \; L \geq 0, \; K \geq 0
\end{align*}
\]

• The decision variables are: Q, L, K, p?
Firm’s Problem: Description

• In order to determine whether or not $p$ is a decision variable, or whether it depends on $Q$, etc., we need information about the product market:
  
  – Is it competitive? If it is, $p$ is given (is considered like “data” by the firm).
  
  – Does the firm have some market power? If it does, $p$ is not independent of $Q$. 
Cost Minimization

• Now, let us postpone the problem of profit maximization and let us think of the “internal” problem of the firm taking the production level as given: $Q_0$.

• Given $Q_0$, the goal of profit maximization implies, as an intermediate goal, the cost minimization of producing $Q_0$. 
Cost Minimization

\[ \text{Max}_{\{L,K\}} pQ_0 - wL - rK, \text{ being } pQ_0 \text{ a cte} \]

\[ \iff \]

\[ \text{Max}_{\{L,K\}} -(wL + rK) \]

\[ \iff \]

\[ \text{Min}_{\{L,K\}} wL + rK \]
Cost Minimization

• Given $Q_0$, profit maximization requires cost minimization; that is, the firm’s problem is:

$$\min_{\{L,K\}} wL + rK$$
$$s.t. \ Q \geq F(L, K)$$
$$L \geq 0, K \geq 0$$
Cost Minimization

• If $C$ is any cost level, the isocost line $C = wL + rK$ contains all inputs combinations $(L,K)$ which cost $C$ euros. The equation of this straight line is:

$$K = \frac{C}{r} - \frac{w}{r}L$$

$C_2 > C_1 > C_0$

Slope $= \frac{-w}{r}$
Cost Minimization

• Graphically, the solution of the cost minimization problem is:
Cost Minimization

• The solution of the cost minimization problem is the conditional factor demands:
  \[ L^* = L(Q,w,r); \quad K^* = K(Q,w,r) \]

• The Total Cost Function provides the minimum cost of producing \( Q \) at prices \( (w,r) \):
  \[ C(Q,w,r) = wL^* + rK^* = wL(Q,w,r) + rK(Q,w,r) \]

• Average Total Cost and Marginal Cost are:
  \[\text{ATC} = \frac{C(Q,w,r)}{Q} \]
  \[\text{MC} = \frac{dC(Q,w,r)}{dQ} \]
Cost Minimization: Short Run

- In the short run, some of the factors are fixed. Let us suppose in our context that $K=K_0$. Thus, the cost minimization problem is:

\[
\begin{align*}
\min_{\{L\}} & \quad wL + rK_0 \\
\text{s.t.} & \quad Q \leq F(L, K_0) \\
& \quad L \geq 0
\end{align*}
\]

where $rK_0$ is a constant, which will be named Fixed Cost (FC).
Cost Minimization: Short Run

- With only two inputs, the cost minimization problem in the short run is trivial:
  \[ F(L, K_0) = Q \rightarrow L^* = L(Q) \]

- In this case, \( L^* \) is independent of \((w, r)\), because given \( K_0 \) the isoquant determines the conditional labour demand.
Cost Minimization: Short Run

• But in general, with more than two inputs, the cost minimization problem in the short run also makes sense.

• Let us suppose \( L = (L_1, L_2) \); \( K_0 = (K_{01}, K_{02}) \), where \( K_0 = (K_{01}, K_{02}) \) is a vector of constants.

• The problem in this case will be:

\[
\begin{align*}
\min_{\{L_1, L_2\}} & \quad w_1 L_1 + w_2 L_2 + r_1 K_{01} + r_2 K_{02} \\
\text{s.t.} & \quad Q \leq F(L_1, L_2, K_{01}, K_{02}) \\
& \quad L_1 \geq 0, L_2 \geq 0
\end{align*}
\]
Cost Minimization: Short Run

- The solution is the conditional factors demands in the short run:
  \[ L_1^* = L_1(Q, w_1, w_2, r_1, r_2); \quad L_2^* = L_2(Q, w_1, w_2, r_1, r_2) \]
Cost Minimization: Short Run

- Let us go back to the two-inputs case, with only one of them variable in the short run.

- The **Total Cost Function** in the short run is:
  
  \[ CT_{SR}(Q,w,r) = wL(Q) + rK_0 , \]

  where \( wL(Q) \) is the variable cost in the short run (\( VC_{SR} \)), and \( rK_0 \) is the fixed cost in the SR (\( FC_{SR} \)).

- **Average Total Cost in the SR:**
  
  \[ ATC_{SR} = TC_{SR}(Q,w,r)/Q = AVC_{SR} + AFC_{SR} , \]

  where \( wL(Q)/Q \) is the average variable cost in the short run, and \( rK_0 /Q \) is the average fixed cost in the short run.
Cost Minimization: Short Run

• Marginal Cost in the SR:
  \[ MC_{SR} = \frac{dTC_{SR}(Q)}{dQ} = \frac{dVC_{SR}(Q)}{dQ} \]

- Remark: the TC\(_{SR}\) is different from the TC in the long run, and therefore ATC is also different. MC\(_{SR}\) does not coincide either with MC in the long run.
Examples: Costs and Returns to Scale

• Let us suppose these three production functions:

  (a) $F(L, K) = LK \uparrow$ increasing returns
  (b) $F(L, K) = \sqrt{LK} \uparrow$ constant returns
  (c) $F(L, K) = \sqrt[3]{LK} \uparrow$ decreasing returns

• For all of them, the cost minimization problem is:

  \[
  \min_{\{L, K\}} wL + rK
  
  \text{s.t.} \quad Q \leq F(L, K)
  
  \text{s.t.} \quad L \geq 0, K \geq 0
  \]
Examples: Costs and Returns to Scale

• In all these three cases we have interior solutions, so solving the problem of the firm is the same as solving the following system:

\[
\begin{align*}
(1) \quad |MRTS(L, K)| &= \frac{w}{r} \\
(2) \quad F(L, K) &= Q
\end{align*}
\]

• MRTS coincides for the three functions. Therefore, condition (1) coincides for all of them:

\[
\frac{K}{L} = \frac{w}{r} \implies K = \left(\frac{w}{r}\right)L
\]
Examples: Costs and Returns to Scale

• Let us calculate the solution for the function (a). Plug $K = (w/r)L$ into the production function:

$$Q = KL = (w/r)L \cdot L = (w/r)L^2$$

• Conditional demands for production factors:

$$L^* = \sqrt{\frac{r}{w}Q}; \quad K^* = \sqrt{\frac{w}{r}Q}$$

• Costs:

$$TC(Q) = wL^* + rK^* = 2\sqrt{wr} \sqrt{Q}$$

$$ATC(Q) = 2\sqrt{wr} \frac{1}{\sqrt{Q}}; \quad MC(Q) = \sqrt{wr} \frac{1}{\sqrt{Q}}$$
Examples: Costs and Returns to Scale

• For \( w=1, \ r=4 \) we get

\[
TC(Q) = 4\sqrt{Q}; \quad ATC(Q) = 4/\sqrt{Q}; \quad MC(Q) = 2/\sqrt{Q}
\]
Examples: Costs and Returns to Scale

• Let us calculate the solution for the function (b). Plug $K = (w/r)L$ into the production function:

$$Q = \sqrt{LK} = \sqrt{(w/r)L^2}$$

• Conditional demands for production factors:

$$L^* = Q\sqrt{\frac{r}{w}}; \quad K^* = Q\sqrt{\frac{w}{r}}$$

• Costs:

$$TC(Q) = wL^* + rK^* = Q2\sqrt{wr}$$

$$ATC(Q) = 2\sqrt{wr}; \quad MC(Q) = 2\sqrt{wr}$$
Examples: Costs and Returns to Scale

- For $w=1$, $r=4$ we get

\[ TC(Q) = 4Q; \quad ATC(Q) = 4; \quad MC(Q) = 4 \]
Examples: Costs and Returns to Scale

• Let us calculate the solution for the function (c). Plug $K = (w/r)L$ into the production function:

$$Q = \sqrt[3]{LK} = \sqrt[3]{(w/r)L^2}$$

• Conditional demands for production factors:

$$L^* = \sqrt[3/2]{\frac{Q}{w}}; \quad K^* = \sqrt[3/2]{\frac{w}{r}}$$

• Costs:

$$TC(Q) = wL^* + rK^* = Q^{3/2} 2\sqrt{wr}$$

$$ATC(Q) = Q^{1/2} 2\sqrt{wr} \quad ; \quad MC(Q) = Q^{1/2} 3\sqrt{wr}$$
Examples: Costs and Returns to Scale

• For $w=1$, $r=4$ we get

$$TC(Q) = 4Q^{3/2}; \quad ATC(Q) = 4Q^{1/2}; \quad MC(Q) = 6Q^{1/2}$$
Reconsidering the Firm’s Problem

The firm’s problem, in the short run or in the long run, can be written as

\[ \max \pi(Q) = TR(Q) - TC(Q) \]
\[ s.t. \quad Q \geq 0 \]

We find the solution using the FOC, and then check the SOC and the Shutting Down Condition:

**FOC:** \[ MR(Q) = MC(Q) \Rightarrow Q^* \]

**SOC:** \[ \pi''(Q) = MR'(Q) - MC'(Q) \leq 0 \]

**SDC:** \[ \pi(Q^*) \geq \pi(0) \]
Reconsidering the Firm’s Problem

Conditions FOC, SOC, SDC provide a solution to the firm’s problem whether it is a competitive firm or not -- for example, when it is a monopoly.

For a competitive firm, the marginal revenue is a constant (the market price). In the monopoly case, however, the marginal revenue will depend on Q.