Market Failure: Public Goods

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Market Failure: Public Goods

Consider a simple economy in which there is one public good (x) and one private good (y).

The public good, of which the economy lacks initially, can be produced using private good at a cost

$$c : \mathbb{R}_+ \to \mathbb{R}_+.$$

Each consumer $i \in \{1, ..., n\}$ is endowed with $\bar{y}_i \in \mathbb{R}_+$ units of the private good, and her preferences for private and public good are described by a utility function

$$u_i : \mathbb{R}^2_+ \to \mathbb{R}.$$

Thus, an the economy is described by the collection

$$[(u_1, \bar{y}_1), ..., (u_n, \bar{y}_n), c].$$

An allocation is a vector (x, y), where $x \in \mathbb{R}_+$ is the amount of public good provided and $y = (y_1, ..., y_n) \in \mathbb{R}^n_+$ is a vector indicating the amount of private good allocated to each consumer.

An allocation (x, y) is *feasible* if

$$\sum_{i=1}^{n} y_i + c(x) \leq \sum_{i=1}^{n} \bar{y}_i.$$

An allocation (x', y') is *Pareto superior* to the allocation (x, y) if the inequality

$$u_i(x', y_i') \geq u_i(x, y_i) \ \forall i, \text{ and } \sum_{i=1}^{n} u_i(x', y_i') > \sum_{i=1}^{n} u_i(x, y_i).$$

An allocation is *Pareto optimal* if it is feasible and there is no *feasible* Pareto superior allocation.

Public Goods

For a profile of *weights* $\lambda = (\lambda_1, ..., \lambda_n) \in \triangle^n$ define the problem $P(\lambda)$ as

$$\max_{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+^n} \sum_{i=1}^n \lambda_i u_i(x, y_i)$$

$$s.t.:$$

$$\sum_{i=1}^n y_i + c(x) \leq \sum_{i=1}^n \bar{y}_i.$$

Proposition. The allocation (x, y) is Pareto optimal if and only if it solves $P(\lambda)$ for some $\lambda \in \triangle^n$.

Proof.

$\Rightarrow$) Let $(x, y_1, ..., y_n)$ be a solution to $P(\lambda)$ for some $\lambda \gg 0$. If there is a feasible Pareto superior allocation $(x', y'_1, ..., y'_n)$, then $\lambda_i u_i(x', y'_i) \geq \lambda_i u_i(x, y_i)$ for all i, and $\lambda_j u_j(x', y'_j) > \lambda_j u_j(x, y_j)$ for some j implies

$$\sum_{i=1}^{n} \lambda_i u_i(x', y'_i) > \sum_{i=1}^{n} \lambda_i u_i(x, y_i),$$

which contradicts that $(x, y_1, ..., y_n)$ is a solution to $P(\lambda)$.

$\Leftarrow$) More involved.

Assume that each u_i is increasing in x and y, differentiable and concave, and that c is differentiable, increasing and convex.

Then the solutions to a problem $P(\lambda)$ are critical points of the Lagrangian:

$$\mathcal{L}(x, y_1, ..., y_n, \mu) = \sum_{i=1}^{n} \lambda_i u_i(x, y_i) + \mu \left(\sum_{i=1}^{n} \bar{y}_i - \sum_{i=1}^{n} y_i - c(x) \right).$$

These critical points are solutions to the system of equations:

$$(x) \qquad \frac{\partial \mathcal{L}}{\partial x} = \sum_{i=1}^{n} \lambda_i \frac{\partial u_i}{\partial x} - \mu c'(x) = 0$$

$$(y_i) \qquad \frac{\partial \mathcal{L}}{\partial y_i} = \lambda_i \frac{\partial u_i}{\partial y_i} - \mu = 0, \ \forall i$$

$$(\mu) \qquad \frac{\partial \mathcal{L}}{\partial \mu} = \sum_{i=1}^{n} \bar{y}_i - \sum_{i=1}^{n} y_i - c(x) = 0.$$

Public Goods

For $\lambda \gg 0$, by equation (y_i) we get

$$\lambda_i \frac{\partial u_i}{\partial y_i} = \mu > 0 \Leftrightarrow \frac{\lambda_i}{\mu} = \frac{1}{\frac{\partial u_i}{\partial y_i}}, \ \forall i.$$

Hence equation (x) may be written as

$$c'(x) = \sum_{i=1}^{n} \left(\frac{\lambda_i}{\mu}\right) \frac{\partial u_i}{\partial x} = \sum_{i=1}^{n} \frac{\frac{\partial u_i}{\partial x}}{\frac{\partial u_i}{\partial y_i}} = \sum_{i=1}^{n} RMS_i(x, y_i).$$

Therefore, a Pareto optimal allocation (x, y) is a feasible allocation, that is, an allocation satisfying equation (μ)

$$\sum_{i=1}^{n} y_i + c(x) = \sum_{i=1}^{n} \bar{y}_i,$$

such that

$$\sum_{i=1}^{n} RMS_i(x, y_i) = c'(x).$$

An Example: A Public Good

Example 1. Consider an economy in which each individual is endowed with 12 hours of time and cares exclusively about her consumption. There is a technology freely available that allows to produce K units of consumption good for each hour of labor used as input. The parameter K represents the *state of knowledge*, and is given by

$$K = \sum_{i=1}^{n} x_i,$$

where x_i is the number of hours individual i spends improving the technology.

Identify the Pareto optimal state of knowledge K^*.

An Example: Pareto Optimality

Let us solve the social welfare problem

$$\max_{(K, y_1, \ldots, y_n)} \sum_{i=1}^{n} u_i(K, y_i), \ s.t. \ K + \sum_{i=1}^{n} y_i = 12n,$$

where y_i is the number of hours individual i uses to produce its own consumption, and K is the total number of hours used to improve the technology.

Using $u_i(K, y_i) = Ky_i$ and $\sum_{i=1}^{n} y_i = 12n - K$ we may write

$$\sum_{i=1}^{n} u_i(K, y_i) = K \sum_{i=1}^{n} y_i = K(12n - K),$$

so that we need to solve the problem

$$\max_{K \geq 0} K(12n - K),$$

whose solution is $K^* = 6n$. That is, the optimal per-capita time allocated to improve the technology is 6 hours.

An Exampe: Voluntary Contributions

Under voluntary contributions an individual decides the time she spends improving the technology by solving the problem

$$\max_{z \geq 0} (K_{-} + z)\,(12 - z)\,,$$

where K_{-} is the total number of hours the other individuals allocate to improving the technology.

The solution to this problem is

$$z^{*} = \frac{12 - K_{-}}{2}.$$

An Exampe: Voluntary Contributions

Let us assume that the (Nash) equilibrium of the (static, i.e., simultaneous) game individuals face is symmetric. (Indeed, it is!) Then

$$z^* = \frac{12 - (n-1)z^*}{2} = \frac{12}{n+1}.$$

Thus, per-capita time allocated to improving the technology is $z^*(n) < 6$ for $n > 1$.

Voluntary contributions leads to under provision of the public good:

This is the Tragedy of the Commons!

Public Goods: Lindahl Equilibrium

Lindahl, observing the dual role of prices and quantities in markets, proposes a "solution" to the *free riding problem*.

The solution involves creating a "market" for public goods in which each individual pays a personalized price.

In the economy described above, a *Lindahl equilibrium* is a collection $(p^*, x^*, y^*) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+$ such that:

(1) $y_i^* = \bar{y}_i - p_i^* x^*$

(2) $c(x^*) = \sum_{i=1}^n p_i^* x^*$

(3) $x^* \in \arg\max u_i(x, \bar{y}_i - p_i^* x), \ \forall i \in \{1, ..., n\}$

(4) $\sum_{i=1}^n p_i^* = c'(x^*)$.

Public Goods: Lindahl Equilibrium

A Lindahl equilibrium allocation is Pareto optimal: By equations (1) and (2), it is feasible,

$$\sum_{i=1}^{n} y_i^* + c(x^*) = \sum_{i=1}^{n} \bar{y}_i - \sum_{i=1}^{n} p_i^* x + c(x^*) = \sum_{i=1}^{n} \bar{y}_i,$$

while by equations (3) and (4)

$$\sum_{i=1}^{n} RMS_i(x^*, y_i^*) = \sum_{i=1}^{n} p_i^* = c'(x^*).$$

Public Goods: Lindahl Equilibrium

Samuelson (1954) argues that while the Lindahl equilibrium is a useful concept (i.e., it identifies allocations satisfying desirable properties, such as

- Pareto optimality, and
- individual rationality),

the idea of setting a market for public goods is unworkable since each individualized market would be a monopsony.

The fundamental issue to settle is how to elicit (i.e., extract the information about) individuals' preferences in order to design the system of personalized prices.

Public Goods: Mechanism Design

The issue raised by Samuelson (1954), that a fundamental part of the problem is that individuals' preferences are unknown, can be posed as a problem of *institution (or mechanism) design*. An earlier literature dealt with this issue framing the problem as a complete information game. (A very strong assumption!)

A mechanism is a pair (S, ϕ) given by

$$S = S_1 \times ... \times S_n,$$

where each S_i is a set of *actions or messages* individual $i \in \{1, ..., n\}$ can choose, and

$$\phi : S \to A$$

is an *outcome function* associating a feasible allocation $\phi(s) \in A \subset \mathbb{R}^n_+ \times \mathbb{R}_+$ to each profile of messages.

Walker (1973)'s mechanism *implements* the Lindahl allocation.

Public Goods: Walker's Mechanism

Consider a simple public good economy as describe above, in which the preferences of individual $i \in \{1, ..., n\}$, where $n > 2$, are represented by a utility function $u_i(x, y_i) = y_i + v_i(x)$, where $v_i : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave. Also, assume that the public good can be produced with constant returns to scale, i.e., $c(x) = \alpha x$, where $\alpha \geq 0$.

Walker's mechanism is given by (S, ϕ), where $S_1 = ... = S_n = \mathbb{R}$, and for $s \in \mathbb{R}^n$,

$$\begin{aligned} \phi_x(s) &= \sum_{j=1}^{n} s_j, \\ \phi_{y_i}(s) &= \bar{y}_i - p_i(s)\phi_x(s), \text{ where } p_i(s) = \frac{\alpha}{n} + (s_{i-1} - s_{i+1}). \end{aligned}$$

(For $i = 1$, we take $i - 1 := n$, and for $i = n$, we take $i + 1 := 1$.)

Example. Ann, Bob and Conrad share an apartment. The apartment has central heating and the temperature can be set at a cost $C(x) = cx$.

Calculate the equilibrium of Walker's mechanism assuming that their preferences for the temperature at the apartment (x) and income are represented by utility functions of the form

$$u_i(x, y_i) = \bar{y}_i - \alpha_i \left(t_i - x\right)^2,$$

where $(\alpha_A, t_A) = (3/2, 25)$, $(\alpha_B, t_B) = (1, 20)$, $(\alpha_C, t_C) = (1, 22)$, for the values of the constant marginal $c \in \{0, 2\}$.

Individual i's problem is:

$$\max_{s\in\mathbb{R}} \ \bar{y}_i - \left(\frac{c}{3} + (s_{i-1} - s_{i+1})\right)(s + s_{i-1} + s_{i+1})$$
$$-\alpha_i\left(t_i - (s + s_{i-1} + s_{i+1})\right)^2.$$

That is

$$-\left(\frac{c}{3} + (s_{i-1} - s_{i+1})\right) + 2\alpha_i(t_i - (s + s_{i-1} + s_{i+1})) = 0,$$

i.e.,

$$s_i = t_i - \frac{c}{3} - \left(1 + \frac{1}{2\alpha_i}\right)s_{i+1} - \left(1 - \frac{1}{2\alpha_i}\right)s_{i-1},$$

where $i-1 = C$ and $i+1 = B$ for $i = A$, $i-1 = A$ and $i+1 = C$ for $i = B$, and $i-1 = B$ and $i+1 = A$ for $i = C$.

Solving the system

$$\begin{aligned} s_A &= 25 - \frac{c}{3} - \left(1 + \frac{1}{3}\right) s_B - \left(1 - \frac{1}{3}\right) s_C \\ s_B &= 21 - \frac{c}{3} - \left(1 + \frac{1}{2}\right) s_C - \left(1 - \frac{1}{2}\right) s_A \\ s_C &= 22 - \frac{c}{3} - \left(1 + \frac{1}{2}\right) s_A - \left(1 - \frac{1}{2}\right) s_B \end{aligned}$$

we get

$$s^* = (s_A^*, s_B^*, s_C^*) = (\frac{25}{3} - \frac{c}{9}, \frac{31}{3} - \frac{c}{9}, \frac{13}{3} - \frac{c}{9})$$

Hence

$$x\left(s^{*}(c)\right)=\frac{25}{3}+\frac{31}{3}+\frac{13}{3}-\frac{3c}{9}=23-\frac{c}{3},$$

and

$$\left(p_A\left(s^{*}(c)\right),p_B\left(s^{*}(c)\right),p_C\left(s^{*}(c)\right)\right)=(-6+\frac{c}{3},4+\frac{c}{3},2+\frac{c}{3}).$$