Midterm (March 19, 2018)

Answer exercise 1, and either exercise 2 or exercise 3. Every part is worth 10 points.

Exercise 1. In an economy that extends over two dates, today and tomorrow, there is a single perishable good, consumption. The state of nature tomorrow can be either sunny or cloudy. There are two consumers whose preferences for consumption today (x), consumption tomorrow if sunny (y), and consumption tomorrow in cloudy (z) are represented by the utility functions $u_1(x, y, z) = xy$, and $u_2(x, y, z) = xz$, and whose endowments are $(\bar{x}_1, \bar{y}_1, \bar{z}_1) = (4, 0, 0)$ and $(\bar{x}_2, \bar{y}_2, \bar{z}_2) = (0, 0, 0)$, respectively. There is a firm that transforms consumption today in consumption tomorrow: regardless of the state, using ω units of consumption today as input the firm produces $2\sqrt{\omega}$ units of consumption tomorrow. The firm is owned by consumer 2.

In part (a) to (c) assume that there are only spot markets and a market for credit. Normalize the spot market prices to 1, i.e., $p_x = p_y = p_z = 1$, and use the notation $R = 1+r$, where r is the interest rate. Note that in this setting, the firm pays profits tomorrow.

(a) Calculate the firm's revenue when sunny and when cloudy as a function of ω and R. Note that it is the same is both states, and so it is its profit. Hence the firm chooses $\omega \geq 0$ to maximize this profit function. Obviously, the firm must borrow to buy input and, since $p_x = 1$, the firm's demand of credit is equal to number of units of input it uses. Calculate the firm's demand of credit, $\omega(R)$, and its profit, $\pi(R)$.

If the firm borrows ω and uses it to buy consumption today to use it as input, then it produces tomorrow an output of $2\sqrt{\omega}$ units in either state. Then, regardless of the state, the firm sells its output (at price 1), rising a revenue of $2\sqrt{\omega}$, and pays its credit, $R\omega = (1+r)\omega$; hence the firm's profit is $2\sqrt{\omega} - R\omega$. Thus, the firm chooses ω to solve the problem

$$\max_{\omega \geq 0} 2\sqrt{\omega} - R\omega.$$

Solving the first order condition for a solution we get the firm's demand of credit, $\omega(R) = 1/R^2$, and its profit $\pi(R) = 1/R$.

(b) Calculate the consumers' demand of credit, $b_i(R)$ for $i \in \{1, 2\}$.

Consumer 1's problem is

$$\max_{(x,y,z)\in\mathbb{R}_+} xy,$$
$$s.t.\ x = 4 + b,\ y = z = -Rb.$$

Since $y = z \geq 0$, then $b \leq 0$. Then we we may write the problem as

$$\max_{b \leq 0} -(4 + b)\, Rb.$$

Solving the first order condition we get consumer 1's demand of credit, $b_1(R) = -2$.

Consumer 2's problem is

$$\max_{(x,y,z)\in\mathbb{R}_+} \quad xz,$$
$$s.t.\ x = b,\ \ y = z = \pi(R) - Rb.$$

Since $y = z \geq 0$*, then* $b \geq 0$*. Then, using* $\pi(R) = 1/R$*, we may write the problem as*

$$\max_{b\geq 0} b\left(\frac{1}{R} - Rb\right).$$

Solving the first order condition we get consumer 2's demand of credit, $b_2(R) = 1/\left(2R^2\right)$.

(c) Calculate the competitive equilibrium (CE) interest rate and allocation. Is the CE allocation Pareto optimal?

The credit market clearing condition is

$$\omega(R) + b_1(R) + b_2(R) = 0 \Leftrightarrow \frac{1}{R^2} - 2 + \frac{1}{2R^2} = 0.$$

Solving for R *we get* $R^* = \sqrt{3}/2$.

The equilibrium allocation is $\omega^* = \omega(R^*) = 4/3$*, and* $(x_1^*, y_1^*, z_1^*) = (2, \sqrt{3}, \sqrt{3})$, $(x_2^*, y_2^*, z_2^*) = (2/3, 1/\sqrt{3}, 1/\sqrt{3})$*. Obviously this allocation is not Pareto optimal, since the allocation* $\omega = 4/3$, $(x_1, y_1, z_1) = (2, 2/\sqrt{3}, 0)$, $(x_2, y_2, z_2) = (2/3, 0, 2/\sqrt{3})$ *is Pareto superior.*

In parts (d) to (f) assume that there are spot markets each date, and markets for two securities, Y and Z, that operate today. A unit of security Y pays one unit of consumption tomorrow if sunny and nothing if cloudy, and a unit of security Z pays one unit of consumption tomorrow if cloudy and nothing if sunny. No credit market exists. Normalize the spot market prices to 1, i.e., $p_x = p_y = p_z = 1$, and use the notation q_Y and q_Z for the prices of the securities Y and Z, respectively.

(d) In this setting, the firm buys ω units of good x to use as input, and sells its output tomorrow when sunny and when cloudy $(2\sqrt{\omega})$ in the markets for security Y and Z, respectively. Solve the firm's profit maximization problem to calculate its demand of good x, $\omega(q_Y, q_Z)$, and its supply of securities Y and Z, $Y_f(q_Y, q_Z)$, and $Z_f(q_Y, q_Z)$. Also calculate the firm's profits $\pi(q_Y, q_Z)$, which in this setting are realized today (which is when all transactions occur).

If the firm uses ω *units of good* x *as input, then it produces tomorrow an output of* $2\sqrt{\omega}$ *units of consumption in either state, which it sells in the market for securities* Y *and* Z*. Hence its cost is* ω *(since* $p_x = 1$*) and its revenue is* $(q_Y + q_Z)2\sqrt{\omega}$*. Thus, the firm solves the problem*

$$\max_{\omega\geq 0} \ 2\left(q_Y + q_Z\right)\sqrt{\omega} - \omega.$$

Solving the first order condition for a solution we get the firm's demand of good x *is* $\omega(q_Y, q_Z) = (q_Y + q_Z)^2$*, its supply of securities is* $Y_f(q_Y, q_Z) = Z_f(q_Y, q_Z) = 2(q_Y + q_Z)$*, and its profit is* $\pi(q_Y, q_Z) = (q_Y + q_Z)^2$.

(e) Calculate the consumers' demands of securities Y and Z, $Y_i(q_Y, q_Z)$ and $Z_i(q_Y, q_Z)$ for $i \in \{1, 2\}$.

Consumer 1's problem is

$$\begin{aligned} &\max_{(x,y,z)\in\mathbb{R}_+} xy, \\ &s.t.\ x + q_Y Y + q_Z Z = 4,\ y = Y,\ z = Z. \end{aligned}$$

Hence $z = Z = Z_1(q_Y, q_Z) = 0$, *and using* $x = 4 - q_Y Y$ *we may write the problem as*

$$\max_{Y \geq 0} (4 - q_Y Y)\, Y.$$

Solving the first order condition we get $Y_1(q_Y, q_Z) = 2/q_Y$.

Consumer 2's problem is

$$\begin{aligned} &\max_{(x,y,z)\in\mathbb{R}_+} xz, \\ &s.t.\ x + q_Y Y + q_Z Z = \Pi(q_Y, q_Z),\ y = Y,\ z = Z. \end{aligned}$$

Hence $y = Y = Y_2(q_Y, q_Z) = 0$, *and using* $x = \Pi(q_Y, q_Z) - q_Z Z = (q_Y + q_Z)^2 - q_Z Z$ *we may write the problem as*

$$\max_{Z \geq 0} \left((q_Y + q_Z)^2 - q_Z Z\right) Z.$$

Solving the first order condition we get $Z_2(q_Y, q_Z) = (q_Y + q_Z)^2 / (2q_Z)$.

(f) Calculate the CE security prices and allocation. Is the CE allocation Pareto optimal?

The security markets clearing conditions are

$$\begin{aligned} Y_f(q_Y, q_Z) &= Y_1(q_Y, q_Z) + Y_2(q_Y, q_Z) \Leftrightarrow 2(q_Y + q_Z) = \frac{2}{q_Y} \\ Z_f(q_Y, q_Z) &= Z_1(q_Y, q_Z) + Z_2(q_Y, q_Z) \Leftrightarrow 2(q_Y + q_Z) = \frac{(q_Y + q_Z)^2}{2q_Z} \end{aligned}$$

Solving we get $(q_Y, q_Z) = (\sqrt{3}/2, \sqrt{3}/6)$.

The equilibrium allocation is $\omega^* = 4/3$, *and* $(x_1^*, y_1^*, z_1^*) = (2, 4/\sqrt{3}, 0)$, $(x_2^*, y_2^*, z_2^*) = (2/3, 0, 4/\sqrt{3})$. *Obviously this allocation is Pareto optimal because the price vector* $(p_x^*, p_y^*, p_z^*) = (1, \sqrt{3}/2, \sqrt{3}/6)$ *together with the given allocation forms an Arrow-Debreu equilibrium.*

Exercise 2. Consider a market for lemons in which there is a measure 1 of risk-neutral buyers and sellers. The quality of a seller's good is indexed by his cost. Sellers' costs are uniformly distributed in the interval $[0,1]$. A buyer's value of a unit of quality $x \in [0,1]$ is $x+v$, where $v \in (0,1/2)$. Each seller knows the quality of the good he sells, but quality is not observable to buyers prior to purchase.

(a) Calculate and graph the market supply, and determine the average quality of the cars offered at each price.

At each price p only sellers with qualities $x \leq p$ supply. Then the supply is

$$S(p) = p.$$

Since the random variable $X|\, X \leq p$ is uniformly distributed in $[0,p]$, the expected quality of unit supplied at p is

$$\mathbb{E}(X|\, X \leq p) = p/2.$$

(Here's a formal derivation: Let's first calculate the cumulative distribution function of the random variable $X|\, X \leq p$. If $x \leq p$, then

$$F_{X|X\leq p}(x) = \Pr(X \leq x|\, X \leq p) = \frac{\Pr(X \leq x, X \leq p)}{\Pr(X \leq p)} = \frac{F(x)}{F(p)} = \frac{x}{p}$$

and if $x > p$, then $F_{X|X\leq p}(x) = 0$. Thus, the density function of $X|\, X \leq p$ is $f_{X|X\leq p}(x) = F'_{X|X\leq p}(x) = 1/p$ if $x \leq p$ and $f_{X|X\leq p}(x) = 0$ if $x > p$. Therefore

$$\mathbb{E}(X|\, X \leq p) = \int_{-\infty}^{\infty} x f_{X|X\leq p}(x)dx = \int_0^p \frac{x}{p}dx = \frac{p}{2}.)$$

(b) Calculate and graph the market demand and determine the price, total quantity traded and total surplus realized at the competitive equilibrium.

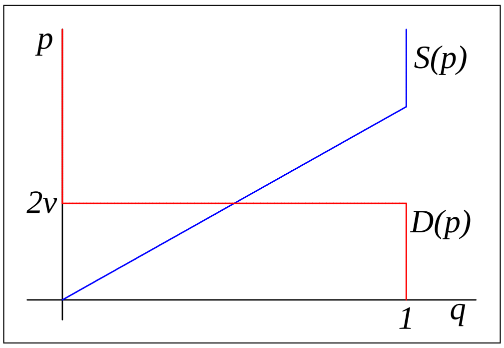
A buyer's expected value of a random unit supplied at price p is

$$\mathbb{E}(X|\, X \leq p) + v = \frac{p}{2} + v.$$

Hence a buyer demands one unit of the good if $p/2 + v > p$, that is, $p < 2v$, demands 0 units if $p > 2v$, and is indifferent between demanding 1 or 0 units if $p = 2v$. Hence the demand is

$$D(p) = \begin{cases} 0 & \text{if } p > 2v \\ [0,1] & \text{if } p = 2v \\ 1 & \text{if } p < 2v. \end{cases}$$

Here's a graph of the supply and demand schedules.



Market clearing $S(p) = D(p) = q^*$ implies $p^* = q^* = 2v < 1$. Since the surplus realized for each unit sold is v, the total surplus is $W = q^*v = 2v^2 < v$. As v approaches $1/2$, total surplus approaches $1/2 = v$.

(c) Calculate the effect on the CE price and total surplus of a subsidy $s \in \mathbb{R}_+$ to each buyer who acquires a unit. Identify the smallest subsidy that maximizes the surplus generated by the units traded. (Hint. You need to consider separately the impact of a subsidy $s \leq 1/2 - v$ and $s > 1/2 - v$.)

With a subsidy $s \leq 1/2 - v$, the CE price is

$$p^*(s) = q^*(s) = 2(v + s),$$

and the total surplus, which is captured by sellers, is

$$W(s) = q^*(s)(v + s) = 2(v + s)^2.$$

Hence the subsidy increases the (sellers) surplus by

$$W(s) - W(0) = 2(v + s)^2 - 2v^2 = 2(v + s)s + 2sv.$$

The first term in the surplus increment due to the direct subsidy, while the second term is the increment of surplus generated by the additional $2s$ units that are traded when the subsidy is introduced.

For $s > 1/2 - v$, there are multiple CE prices, $p^* \in [1, 2(v + s)]$, but the quantity traded in all these CE is $q^*(s) = 1$. Hence the total surplus, which distribution between buyers and sellers is determined by the equilibrium price, is

$$W(s) = q^*(s)(v + s) = v + s.$$

Hence

$$W(s) - W(0) = s + (v - 2v^2),$$

that is, increments of the subsidy above $1/2 - v$ increase the traders' surplus by s, and are a mere transfer which does not change the surplus generated by the market.

Hence the optimal subsidy is $s^* = 1/2 - v$.

Exercise 3. In a typical day, tourists traveling to a city known to be a pickpocket's playground face the risk of loosing the money they carry in their wallet. For the more alert tourists, this happens with probability $p_L = 1/4$, while for the inattentive ones this probability is $p_H = 1/2$. Each tourist carries $W = 150$ euros in the wallet for expenses, and the typical loss is $L = 100$. Tourists preferences are described by the von Neumann-Morgenstern utility function $u(x) = \ln x$.

(a) Assume that there is a competitive insurance market where tourist may subscribe a policy covering this risk. Determine the policies that will be offered assuming that insurance companies can tell whether a tourist is of the alert or the inattentive type.

Since the market is competitive, under complete information companies will offer the fair premium full insurance policy to each type; that is, they will offer the policy

$$(I_H, 0) = (100p_H, 0) = (50, 0)$$

to the inattentive tourists, and the policy

$$(I_L, 0) = (100p_L, 0) = (25, 0)$$

to the alert tourists.

(b) Assume now that insurance companies cannot tell whether a tourist subscribing a policy is of the alert or the inattentive type, and that there are twice as many alert than inattentive tourists. Which insurance policies will be offered? (To solve an equation you will encounter, these formulae is useful: $a \ln x + b \ln y = \ln(x^a y^b)$; $ax^2 + bx + c = 0 \Leftrightarrow x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$.)

As established in class, a competitive equilibrium, when it exists, offers separating fair policies $(I_H, 0) = (50, 0)$ *and* $(\hat{I}_L, \hat{D}_L)$ *such that*

$$\hat{I}_L = (100 - \hat{D}_L)p_L,$$

and such that the inattentive tourists are indifferent between the two policies, i.e.,

$$\frac{1}{2}\ln\left(150 - (100 - \hat{D}_L)p_L - \hat{D}_L\right) + \frac{1}{2}\ln\left(150 - (100 - \hat{D}_L)p_L\right) = \ln(150 - 50).$$

This equation may be written for $x = D_L$ *as*

$$(150 - (100 - x)/4 - x)(150 - (100 - x)/4) = (100)^2.$$

Solving this equation we get

$$\hat{D}_L = 100(2\sqrt{13} - 5)/3 \simeq 73.703.$$

Hence

$$\hat{I}_L = 200(4 - \sqrt{13})/9 \simeq 8.765.$$

For these policies to form a competitive equilibrium the alert tourist must prefer the policy $(\hat{I}_L, \hat{D}_L)$ *to the pooling policy* $(\bar{I}, 0) = (100\bar{p}, 0)$*, where*

$$\bar{p} = \frac{1}{3}p_H + \frac{2}{3}p_L = \frac{1}{3}.$$

That is $(\bar{I}, 0) = (100/3, 0)$*. The expected utility of an alert tourist with the policy* $(\hat{I}_L, \hat{D}_L)$ *is*

$$\frac{1}{4}\ln\left(150 - \left(200(4-\sqrt{13})/9\right) - \left(100(2\sqrt{13}-5)/3\right)\right) + \frac{3}{4}\ln\left(150 - \left(200(4-\sqrt{13})/9\right)\right) \simeq 4.766,$$

and his expected utility with the pooling policy $(\bar{I}, 0)$ *is*

$$\ln(150 - 100/3) \simeq 4.759.$$

Hence the policies $(I_H, 0)$ *and* $(\hat{I}_L, \hat{D}_L)$ *form a competitive equilibrium in this market.*

(c) Assume that the market is monopolized by a single company, which by law must offer a single insurance policy to all tourists. (That is, the firm cannot "screen" tourists with a menu of policies.) Which policy will this company offer? (Hint. Should the firm offer full insurance? Should it offer a policy intended for both types of tourists, or a policy that attracts only inattentive tourist?) Determine which tourists win and loose in this situation relative to that of part (b).

The company must decide whether to offer a policy that only inattentive tourist subscribe or one which both types of types of tourists subscribe. Obviously, in either case the company will offer full insurance since it can extract more surplus from the risk averse tourists.

If the firm offers a policy that both types subscribe, it has to offer it at the maximum premium the alert tourist are willing to pay, that is,

$$\ln(150 - x) = \frac{1}{4}\ln(150 - 100) + \frac{3}{4}\ln(150),$$

Solving this equation we get $I_L^M = 150 - \left((50)(150)^3\right)^{\frac{1}{4}} \simeq 36$*. The monopoly's expected profit per tourist is*

$$I_L^M - \bar{p}L = 36 - \frac{1}{3}(100) = 2.67.$$

(The probability that the average tourist suffers the loss given is $\bar{p}$*.)*

If the firm offers a policy that only inattentive tourists subscribe, then it charges the premium that solves the equation

$$\ln(150 - x) = \frac{1}{2}\ln(150 - 100) + \frac{1}{2}\ln(150).$$

Solving this equation we get $I_H^M = 150 - 50\sqrt{3} \simeq 63.397$*. Since only* $1/3$ *of the tourists (the inattentive one) subscribe this policy the monopoly's expected profit per tourist is*

$$\frac{1}{3}\left(I_H^M - p_H L\right) = \frac{1}{3}\left(\left(150 - 50\sqrt{3}\right) - \frac{1}{2}(100)\right) \simeq 4.47.$$

Hence the monopoly will offer the policy $\left(I_H^M, 0\right)$.