Midterm – April 2017

Exercise 1. Consider an economy that extends over two periods, today and tomorrow, and where there are two consumers, A and B, and a single perishable good. The state of nature tomorrow can either be sunny (S) or cloudy (C). Consumers' preferences over consumption today (x_0), and tomorrow when sunny (x_S) and cloudy (x_C) are represented by the utility functions $u^A(x_0, x_S, x_C) = x_0 x_S^2 x_C$, and $u^B(x_0, x_S, x_C) = x_0 x_S x_C^2$. Both A and B are endowed with twelve units of the good in each of the two periods.

(a) (30 points) Assume that there are contingent markets for all commodities. Calculate the competitive equilibrium prices. (Hint. Normalize $p_0 = 1$. Calculate consumer i's demands using the condition that his marginal rate of substitution for x_0 and x_τ, $\tau \in \{S, C\}$,

$$MRS^i_{0\tau}(x_0, x_S, x_C) = -\frac{\partial u^i/\partial x_0}{\partial u^i/x_\tau},$$

must be equal to the price ratio $-1/p_\tau$. Then use market clearing conditions to calculate the equilibrium prices (p^*_S, p^*_C).)

Solution: For prices (p_S, p_C), the problem of consumer $i \in \{A, B\}$ is

$$\begin{array}{l} \max_{(x_0,x_S,x_C)\in\mathbb{R}^3_+} u^A(x_0, x_S, x_C) \\ \text{subject to:} \\ x_0 + p_S x_S + p_C x_C \leq 12 + 12p_S + 12p_C. \end{array}$$

The solution to this is interior and solves the system

$$\begin{aligned} MRS^i_{0S}(x_0, x_S, x_C) &= -\frac{1}{p_S} \\ MRS^i_{0C}(x_0, x_S, x_C) &= -\frac{1}{p_C} \\ x_0 + p_S x_S + p_C x_C &= 12 + 12p_S + 12p_C. \end{aligned}$$

Hence for consumer A we have

$$\begin{aligned} -\frac{x_S^2 x_C}{2x_0 x_S x_C} &= -\frac{1}{p_S} \\ -\frac{x_S^2 x_C}{x_0 x_S^2} &= -\frac{1}{p_C} \\ x_0 + p_S x_S + p_C x_C &= 12(1 + p_S + p_C). \end{aligned}$$

That is,

$$\begin{aligned} p_S x_S &= 2x_0 \\ p_C x_C &= x_0 \\ x_0 + p_S x_S + p_C x_C &= 12(1 + p_S + p_C) \end{aligned}$$

Solving the system we get

$$\begin{aligned} x_0^A(p_S, p_C) &= 3(1 + p_S + p_C) \\ x_S^A(p_S, p_C) &= \frac{6(1 + p_S + p_C)}{p_S} \\ x_C^A(p_S, p_C) &= \frac{3(1 + p_S + p_C)}{p_C}. \end{aligned}$$

Symmetrically

$$\begin{aligned} x_0^B(p_S, p_C) &= 3(1 + p_S + p_C) \\ x_S^B(p_S, p_C) &= \frac{3(1 + p_S + p_C)}{p_S} \\ x_C^B(p_S, p_C) &= \frac{6(1 + p_S + p_C)}{p_C}. \end{aligned}$$

Market clearing conditions are

$$\begin{aligned} x_0^A(p_S, p_C) + x_0^B(p_S, p_C) &= 12 + 12 \\ x_S^A(p_S, p_C) + x_S^B(p_S, p_C) &= 12 + 12, \end{aligned}$$

that is

$$\begin{aligned} 6(1 + p_S + p_C) &= 24 \\ \frac{9(1 + p_S + p_C)}{p_S} &= 24. \end{aligned}$$

Solving this system we get

$$(p_S^*, p_C^*) = (\frac{3}{2}, \frac{3}{2}).$$

The equilibrium allocation is

$$\begin{aligned} (x_0^A, x_S^A, x_C^A) &= (12, 16, 8) \\ (x_0^B, x_S^B, x_C^B) &= (12, 8, 16). \end{aligned}$$

Of course, this allocation is Pareto optimal by the First Welfare Theorem.

(b) (25 points) Suppose that there are no contingent markets, by there is a credit market and a market for an security that pays one unit of consumption tomorrow if sunny and 2 units of consumption if cloudy. Determine the competitive equilibrium interest rate r^* and security price q^*. (Hint. You need not repeat all calculations, but explore the relation between (r^*, q^*) and the equilibrium prices found in part (a), (p_S^*, p_C^*).)

Solution: Let us normalize the spot prices to be $(\hat{p}_0, \hat{p}_S, \hat{p}_C) = (1, 1, 1)$. *For* (r, q), *the problem of consumer is* $i \in \{A, B\}$

$$\begin{array}{l} \max_{[(x_0, x_S, x_C), b, y] \in \mathbb{R}^3_+ \times \mathbb{R} \times \mathbb{R}} u^i(x_0, x_S, x_C) \\ \text{subject to:} \\ \qquad x_0 + qy \leq 12 + b \\ \qquad x_S \leq 12 - (1+r)b + y \\ \qquad x_C \leq 12 - (1+r)b + 2y. \end{array}$$

Since consumers' utility functions are strictly increasing in all the arguments, the budget constraints are binding at the solution. Hence, solving for b *and* y *in the equation describing the constraints, we may write the problem as*

$$\begin{array}{l} \max_{(x_0, x_S, x_C) \in \mathbb{R}^3_+} u^i(x_0, x_S, x_C) \\ \text{subject to:} \\ x_0 + (\frac{2}{1+r} - q)x_S + (q - \frac{1}{1+r})x_C \leq 12\left(1 + (\frac{2}{1+r} - q) + (q - \frac{1}{1+r})\right). \end{array}$$

This problem is identical to that of part (a). In equilibrium (r^*, q^*) *solves the system*

$$\begin{aligned} \frac{2}{1+r} - q &= p_S^* \\ q - \frac{1}{1+r} &= p_C^*. \end{aligned}$$

Solving the system we get

$$(q^*, r^*) = \left(\frac{9}{2}, -\frac{2}{3}\right).$$

And of course, the resulting allocation is that of part (a).

Exercise 2. The preferences for weekly leisure (x, measured in days) and income (y, measured in euros) of the 10 inhabitants of Manchester by the Sea are represented by the utility function $u(x,y) = 2x + y$. In this fishermen's village the inhabitants only source of weekly income is the revenues they obtain from selling fish in the local market, in which the price of fish is 2 euros a pound. The pounds of fish caught weekly by each individual i (q_i) depends on how many days he devotes to fishing (z_i), as well as the total number of days all inhabitants devote to fishing, according to the function $q_i = \max\{(23 - z_i - \bar{z}_{-i})z_i, 0\}$, where $\bar{z}_{-i} = \sum_{j \neq i} z_j$.

(a) (15 points) Calculate how many (of the seven) days will each inhabitant devote weekly to leisure and fishing activities. (You may assume without proof that equilibrium is interior and symmetric.) Is the resulting allocation Pareto optimal?

Solution: In order to choose z_i an individual solves the problem

$$\begin{aligned}
&\max\nolimits_{(x,y,z_i)\in\mathbb{R}^3_+} 2x + y \\
&\textit{subject to:} \\
&x = 7 - z_i \\
&y = 2(23 - z_i - \bar{z}_{-i})z_i.
\end{aligned}$$

This problem is equivalent to

$$\max_{z_i \in [0,7]} 2\,(7 - z_i) + 2(23 - z_i - \bar{z}_{-i})z_i.$$

An interior solution solves the equation

$$-2 + 2(23 - 2z_i - \bar{z}_{-i}) = 0,$$

that is,

$$z_i = 11 - \frac{\bar{z}_{-i}}{2}.$$

Hence equilibrium is symmetric, i.e., $z_i = z^*$ and $\bar{z}_{-i} = (n-1)\,z^* = 9z^*$, and therefore the solution to the equation

$$z^* = 11 - \frac{9z^*}{2}$$

is

$$z^* = 2.$$

Thus, each inhabitants devotes two days a week to fishing, and hence enjoys $x^* = 5$ days of leisure, and obtains a revenue $y^* = 2(23 - z^* - 9z^*)z^* = 12$ euros. This allocation of time is no Pareto optimal. For example, if everyone reduces its fishing to just one day, i.e., $z_i = 1$, then each inhabitant bundle would be $(x,y) = (6,26) > (x^*, y^*)$, thus making everybody better off.

(b) (15 points) Find a Pareto optimal allocation that treats equally all inhabitants.

Solution. We may identify an equal treatment Pareto optimal allocation $(x_i, y_i) = (\hat{x}, \hat{y})$ *by maximizing the welfare of the representative inhabitant:*

$$\begin{aligned}&\max_{(x,y,z)\in R^3_+} 2x + y\\ &\text{subjet to:}\\ &x = 7 - z\\ &y = 2(23 - 10z)z.\end{aligned}$$

This problem is equivalent to

$$\max_{Z\in[0,70]} 2\,(7 - z) + 2(23 - 10z)z.$$

The solution to this problem solves the equation

$$-2 + 2(23 - 20z) = 0.$$

That is,

$$\hat{z} = \frac{11}{10} = 1.1,$$

and therefore

$$\hat{x} = 7 - \hat{z} = 5.9$$

and

$$\hat{y} = 2(23 - 10\hat{z})\hat{z} = 26.4.$$

(c) (15 points) Assume that the fishermen association decides to establish a *daily fee* p to be paid by any inhabitant who wants to fish in the coastal area. The revenues from this fee are then distributed equally among the 10 inhabitants. What would be the optimal fee (that is, the Lindahl price)?

With a daily fee p*, an inhabitant's income as a function of the days he devotes to fishing is*

$$\begin{aligned} y_i &= 2(23 - z_i - \bar{z}_{-i})z_i - pz_i + \frac{1}{10}p(z_i + \bar{z}_{-i})\\ &= 2(23 - z_i - \bar{z}_{-i})z_i - \frac{9}{10}pz_i + \frac{p}{10}\bar{z}_{-i}\end{aligned}$$

and his problem becomes

$$\max_{z_i\in[0,7]} 2\,(7 - z_i) + 2(23 - z_i - \bar{z}_{-i})z_i - \frac{9}{10}pz_i + \frac{p}{10}\bar{z}_{-i}.$$

An interior solution solves the equation

$$-2 + 2(23 - 2z_i - \bar{z}_{-i}) - \frac{9}{10}p = 0,$$

that is,

$$z_i = 11 - \frac{9}{10}p - \frac{\bar{z}_{-i}}{2}.$$

Hence equilibrium is symmetric, i.e., $z_i = \tilde{z}(p)$ *and* $\bar{z}_{-i} = 9\tilde{z}(p)$*, and therefore*

$$\tilde{z}(p) = 2 - \frac{9}{55}p.$$

We want to set up the price so that $\tilde{z}(p) = 11/10$*, that is*

$$p^* = \frac{11}{2} = 5.5.$$

With this price the resulting allocation would that identified in part (b).