Midterm – March 2016

Exercise 1. Audry has just graduated from high school, and is now deciding how much more to invest in her education. With no further investment her consumption would be $\bar{x}_1^A = 38$ during her youth, and $\bar{x}_2^A = 16$ during her elderly. However, by investing in education $z \in [0, 16]$ units of her consumption during her youth, she can increase her consumption in her elderly by $f(z) = 4z - z^2/8$ additional units. Her preferences for consumption streams are described by the utility function $u^A(x_1, x_2) = 3 \ln x_1 + 2 \ln x_2$.

(a) (15 points) How much would Audry invest in education if she has no access to credit markets. (Hint. When you set up Audry's problem and derive the first order condition for an interior solution, try the value $z = 8$.) Provide a diagram showing Audry's set of *feasible* consumption streams (x_1, x_2) in $\mathbb{R}^2_+$, and identify her optimal choice.

(b) (20 points) Now assume that Audry has access to credit provided by her parents. Calculate her investment in education as a function of the interest rate at which she can borrow from her parents, r. How much would Audry invest and what would be her consumption stream if she borrows from her parents at no interest (i.e., $r = 0$)? (Hint. Set up Audry's problem and calculate the first order conditions for an interior solution. Denote by b the amount that Audry borrows from her parents. I you inspect these equations you will see that you can easily solve for z as a function of r. It is more cumbersome to solve for b as a function of r, but you need not do this. For the second part of the question, you just need to calculate $z(0)$, then replace this value in one of the equations to calculate $b(0)$, and then calculate Audry's consumption stream.)

(c) (15 points) Now assume that Audry cannot (or refuses to) borrow from her parents, but has access to a competitive credit market in which a single other person, Bondi, participates. Bondi, who did not graduate from high school and hence cannot further invest in education, has an endowment stream equal to $(\bar{x}_1^B, \bar{x}_2^B) = (25, 10)$, and his preferences are represented by the utility function $u^B(x_1, x_2) = x_1 x_2$. Verify that $r^* = 1$ is a competitive equilibrium of the economy formed by Audry and Bondi, and calculate the resulting allocation. (Hint. Here you need to calculate how much Bondi borrows $b^B(r)$; then you can use your results in part (b) to calculate how much Audry borrows for $r = 1$; and then verify that for $r = 1$ Audry's and Bondi's demand of credit clear the market. Finally, you can proceed to calculate Audry's and Bondi's consumption streams.)

Solution: (a) Audry's problem is:

$$\max_{(x_1,x_2,z)\in\mathbb{R}^2_+\times[0,16]} 3\ln x_1 + 2\ln x_2$$

$$\begin{aligned} x_1 &\leq 38 - z \\ x_2 &\leq 16 + f(z). \end{aligned}$$

Since the budget constraints are binding at every solution (because u^A is strictly increasing in both arguments) this problem is equivalent to

$$\max_{z\in[0,16]} 3\ln(38-z) + 2\ln(16+f(z))$$

First order condition for an interior solution is:

$$\frac{3}{38-z} = \frac{2f'(z)}{16+f(z)}$$

i.e.,

$$\frac{3}{38-z} = \frac{2\left(4-\frac{z}{4}\right)}{16+4z-z^2/8}$$

Hence $z^* = 8$.

(b) Audry's problem is now

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} \ln x_1 + 2\ln x_2$$

$$\begin{aligned} x_1 &= 38 + b - z \\ x_2 &= 16 - (1+r)\,b + f(z). \end{aligned}$$

This problem is equivalent to the problem

$$\max_{(b,z)\in\mathbb{R}\times[0,16]} 3\ln(38+b-z) + 2\ln(16-(1+r)\,b+f(z))$$

First order conditions for an interior solution are:

$$\begin{aligned} \frac{3}{38+b-z} &= \frac{2(1+r)}{16-(1+r)\,b+f(z)} \\ \frac{3}{38+b-z} &= \frac{2f'(z)}{16-(1+r)\,b+f(z)} \end{aligned}$$

Hence $f'(z) = 1+r$, *or* $4 - z/4 = 1+r$. *Solving for* z *we get*

$$z(r) = 12 - 4r.$$

For $r = 0$, $z(0) = 12$, *and since* $f'(12) = 1$, $f(12) = 30$. *Substituting these values in either equation above we get* $b(0) = \frac{86}{5} = 17.2$. *Using the budget equation we calculate Audry's consumption stream:*

$$\begin{aligned} x_1^A(0) &= 38 + b(0) - z(0) = 38 + 17.2 - 12 = 43.2 \\ x_2^A(0) &= 16 - (1+0)b(0) + f(12) = 16 - 17.2 + 30 = 28.8. \end{aligned}$$

(It is somewhat tedious, but not difficult, to replace $z(r)$ and $f(z(r))$ in the equation

$$\frac{3}{38+b-z(r)}=\frac{2}{16-(1+r)b+f(z(r))},$$

and solve for b to get

$$b(r)=\frac{86-6r^2-20r}{3r+5}.$$

But this is not needed to answer the question.)

(c) We calculate Bondi's demand of credit. Bondi's problem is

$$\max_{((x_1,x_2),b)\in\mathbb{R}^2_+\times\mathbb{R}} x_1x_2$$
$$\begin{aligned} x_1 &\leq 25+b \\ x_2 &\leq 10-(1+r)b \end{aligned}$$

That is

$$\max_{b\in\mathbb{R}}(25+b)\,(10-(1+r)b)$$

Hence first order condition for a solution is$(y-(1+r)b)-(1+r)\,(x+b)=0$, $b=\frac{1}{2r+2}\,(10-20\,(r+1))=-\frac{10r+5}{1+r}=$

$$(10-(1+r)b)-(1+r)\,(25+b)=0.$$

Solving for b we get

$$b^B(r)=-\frac{1}{2}\frac{25r+15}{1+r}.$$

For $r=1$, we have $b^B(1)=-10$, and $b^A(1)=7.5$, which is not an equilibrium.

(The hint was not meant to be misleading. I simply made a mistake copying Bondi's endowment stream, which I meant it to be $(\hat{x}^B_1,\hat{x}^B_2)=(20,10)$. With this endowment Bondi's demand of credit is $\hat{b}^B(r)=-(10r+5)\,/\,(1+r)$, so that $\hat{b}^B(1)=-7.5$ and therefore $r=1$ is a competitive equilibrium. With Bondi's endowment equal to $(\bar{x}^B_1,\bar{x}^B_2)=(25,10)$ the equilibrium equation is a bit messy to solve. A numerical solution is $r^*\simeq 0.72$. At this interest rate Audry and Bondi borrow $b^A(r^*)=-b^B(r^*)\simeq 9.59$, Audry invests $z^A(r^*)=9.12$, and the equilibrium consumption streams are $\left(x^A_1,x^A_2\right)=(38.47,25.58)$, and $\left(x^B_1,x^B_2\right)=(28.41,26.5)$. But I understand these are messy calculations.)

Exercise 2. Consider an insurance market in which all individuals have the same initial wealth, $W = 51/32$ monetary units, and the same preferences, represented by the von Neumann-Morgenstern utility function $u(x) = \ln x$, where x is the individual's disposable wealth. Each individual faces the risk of loosing one monetary unit (i.e., $L = 1$). For a fraction $\lambda^H \in (0,1)$ of the individuals the probability of suffering this loss is $p^H = 1/2$ whereas for the remaining fraction $\lambda^L = 1 - \lambda^H$ it is $p^L = 1/4$. Insurance companies have this information, but the probability with which a particular individual may suffer the loss is the individual's private information.

(a) (10 points) Calculate the policies that will be offered in the competitive equilibrium. (You will need to calculate the deductible of a certain policy; in your calculations, try the value 3/4.)

(b) (15 points) Suppose that the government puts to a referendum a law imposing to everyone mandatory full coverage insurance, and forbids insurance companies offering other policies. Calculate this policy. (Note that insurance companies would compete to supply this policy.) identify the values of $\lambda^H \in (0,1)$ for which such a proposal would be approved by a majority of the electorate. In you arguments, assume that an individual votes in favor of the proposal only if it improves her situation relative to the competitive equilibrium. (Hint. As part of your calculations you should get a key inequality giving you a bound on λ^H. Do not get hang up on this calculation. Use $\bar{\lambda}^H = 0.2$ is an approximately bound, and proceed to conclude and interpret your results.)

Solution: (a) As seen in class, the competitive equilibrium is separating, and involves offering the policies $(I^H, D^H) = (p^H L, 0) = (1/2, 0)$, *and* (I^L, D^L) *satisfying*

$$\begin{aligned} I^L &= p^L(L - D^L) \\ u(W - p^H L) &= (1 - p^H)u(W - I^L) + p^H u(W - I^L - D^L). \end{aligned}$$

Substituting $I^L = p^L(1 - D^L)$ *into the second equation, and using the parameter values and utility function given we get*

$$\ln(\frac{51}{32} - \frac{1}{2}) = \frac{1}{2}\ln(\frac{51}{32} - (1 - D^L)/4) + \frac{1}{2}\ln(\frac{51}{32} - (1 - D^L)/4 - D^L),$$

that is

$$\left(\frac{51}{32} - \frac{1}{2}\right)^2 = \left(\frac{51}{32} - \frac{1 - D^L}{4}\right)\left(\frac{51}{32} - \frac{1 - D^L}{4} - D^L\right).$$

Solving we get $D^L = 3/4$, *and hence* $I^L = (1 - D^L)/4 = 1/16$.

(b) The mandatory full insurance policy is

$$(\bar{I},0)=(\bar{p}L,0)=(\lambda p^{H}+(1-\lambda)\,p^{L},0)=(\frac{1+\lambda}{4},0)$$

Obviously, the fraction λ *of high risk individuals are better off with this policy and will therefore vote in favor. As for the low risk individuals, they will be in favor only if their welfare with this policy* $(\bar{I},0)$ *is greater than with the policy* $(I^{L},D^{L})=(1/16,3/4)$*, that is*

$$u\left(W-\bar{I}\right)>(1-p^{L})u\left(W-I^{L}\right)+p^{L}u\left(W-I^{L}-D^{L}\right),$$

that is

$$\ln\left(\frac{51}{32}-\frac{1+\lambda}{4}\right)>\frac{3}{4}\ln\left(\frac{51}{32}-\frac{1}{16}\right)+\frac{1}{4}\ln\left(\frac{51}{32}-\frac{1}{16}-\frac{3}{4}\right),$$

which may be written as

$$\left(\frac{51}{32}-\frac{1+\lambda}{4}\right)=\left(\frac{51}{32}-\frac{1}{16}\right)^{\frac{3}{4}}\left(\frac{51}{32}-\frac{1}{16}-\frac{3}{4}\right)^{\frac{1}{4}}$$

The largest value of λ *that satisfies this inequality is* $\bar{\lambda}=\frac{43}{8}-\frac{1}{8}\sqrt[4]{25}49^{\frac{3}{4}}\simeq 0.2.$

Hence the policy proposal would be approved if either high risk individuals are a majority, i.e., $\lambda>0.5$*, or if there are minority smaller than* $\bar{\lambda}=0,2,$ *and would not be approved if* $\lambda\in(0.2,0.5)$.

Exercise 3. A town populated by only three citizens must decide the level of a public good (street cleaning services) x it will have. The unit cost of x is 6 euros. Each citizen $i \in \{1,2,3\}$ is endowed with $\bar{y}_i$ euros, and her preferences are described by a utility function of the form $u^i(x,y) = y + 2a_i\sqrt{x}$, where y denotes income (in euros) available to spend on other goods, and $(a_1, a_2, a_3) = (2, 4, 6)$.

(a) (15 points) Calculate the Pareto optimal levels of public good provision, as well as the provision of public good under voluntary contributions.

(b) (10 points) Verify that $m^* = (s_1^*, s_2^*, s_3^*) = (5/3, 2/3, 5/3)$ is a Nash equilibrium of the game induced by the mechanism (S, ϕ) defined by $S_i = \mathbb{R}$ for $i \in \{1,2,3\}$, and for $s \in S$, $\phi(s) = (x, y_1, y_2, y_3)$, where $x = \sum_{i=1}^{3} s_i$, $y_1 = \bar{y}_1 - (2 + s_2 - s_3)\,x$, $y_2 = \bar{y}_2 - (2 + s_3 - s_1)\,x$, and $y_3 = \bar{y}_3 - (2 + s_1 - s_2)\,x$. Identify the resulting allocation. Can you recognize this allocation?

Solution: (a) Since

$$MRS_i(x,y) = \frac{a_i}{\sqrt{x}},$$

then, as seen in class, an interior Pareto optimal allocation must satisfy

$$MRS_1(x,y_1) + MRS_2(x,y_2) + MRS_3(x,y_3) = c,$$

that is

$$\frac{2+4+6}{\sqrt{x}} = 6.$$

Solving this equation we get $x^* = 4$. Thus, in an interior Pareto optimal allocation the level of public good is $x^* = 4$.

Under voluntary contribution, individual i decides its contribution by solving

$$\max_{z_i \geq 0}\ \bar{y}_i - z_i + 2a_i\sqrt{\frac{z_i + z_{-i}}{c}},$$

were z_{-i} is the sum of the contributions of individuals other than i. Hence in an interior solution

$$-1 + \frac{a_i}{c\sqrt{\frac{z_i + z_{-i}}{c}}} = 0.$$

Hence individual i's reaction function is

$$z_i = \max\{\frac{a_i^2}{c} - z_{-i}, 0\}.$$

In a Nash equilibrium (NE),

$$z_1 + z_2 + z_3 \geq 6.$$

Suppose not; then $z_3 < 6 - z_1 - z_2$*, and therefore individual 3 will increase its contribution according to her reaction function. Assume that* $z_1 > 0$*; then* $z_1 = 4/6 - z_2 - z_3 \leq 4/6 - (6 - z_1) = z_1 - 16/3$*, i.e.,* $z_1 = -16/6 < 0$*, a contradiction. Hence in a NE* $z_1 = 0$*. Assume that* $z_2 > 0$*; then* $z_2 = 16/6 - z_1 - z_3 \leq 16/6 - (6 - z_2) = z_2 - 10/3$*, i.e.,* $z_2 = -10/6 < 0$*, a contradiction. Hence in a NE* $z_2 = 0$*. Thus, the unique Nash equilibrium (NE) is* $z_1^{NE} = z_2^{NE} = 0$ *and* $z_3^{NE} = 6$*, and the public good provided is* $x^{NE} = 6/6 = 1$.

(b) With this mechanism individual i *chooses* s_i *by solving*

$$\max_{s_i \in \mathbb{R}} \bar{y}_2 - (2 + s_{i+1} - s_{i-1})(s_i + s_{i+1} + s_{i-1}) + 2a_i\sqrt{s_i + s_{i+1} + s_{i-1}}.$$

Hence s_i *solves*

$$-(2 + s_{i+1} - s_{i-1}) + \frac{a_i}{\sqrt{s_i + s_{i+1} + s_{i-1}}} = 0.$$

That is, individual i*'s reaction function is*

$$s_i = \frac{a_i^2}{(2 + s_{i+1} - s_{i-1})^2} - s_{i+1} - s_{i-1}$$

Since the profile s^* *is a solution to the system*

$$\begin{aligned} s_1 &= \frac{4}{(2 + s_2 - s_3)^2} - s_2 - s_3 \\ s_2 &= \frac{16}{(2 + s_3 - s_1)^2} - s_3 - s_1 \\ s_3 &= \frac{36}{(2 + s_1 - s_2)^2} - s_1 - s_2, \end{aligned}$$

it is a Nash equilibrium of the game.

The equilibrium allocation is $x = 4$, $y_1 = \bar{y}_1 - (2 + 2/3 - 5/3)\,4 = \bar{y}_1 - 4$, $y_2 = \bar{y}_2 - (2 + 5/3 - 5/3)\,4 = \bar{y}_2 - 8$*, and* $y_3 = \bar{y}_3 - (2 + 5/3 - 2/3)\,4 = \bar{y}_3 - 12$*, which is the Lindahl allocation.*