Consumer Theory: Uncertainty

Risk and Uncertainty

The presence of uncertainty implies that the consequences of each alternative are not known in advance, but depend on the realization of events out of the control of the consumer.

Examples of uncertain decisions:

- career choices
- financing a house
- choosing a car or a life insurance policy
- voting for a political candidate
- investment decisions (in assets, education, etc.).

Risk and Uncertainty

In this context,

alternatives are *lotteries*, and

choosing an alternative involves assuming its uncertain consequences.

That is, making a decision involves *betting* on an alternative.

Lotteries

We restrict attention to situations in which the consequences of decisions are monetary gains or losses.

The alternative choicesare therefore *random variables*, referred to as *lotteries,*

$$l = (x,p),$$

where

$$x = (x_1,\ldots,x_n)$$

are *possible payoffs*, and

$$p = (p_1,\ldots,p_n)$$

are *probabilities* with which these payoffs are received.

Lotteries

Example 1. Jorge has a car that needs to be repaired. The cost of repair is uncertain: it is either 300 euros with probability 1/3, or 1,200 euros with probability 2/3. Alternatively, he has been offered a used car at a price of 1,000 euros.

Jorge cannot do without a car as there is no public transportation that he can use for his daily activity.

Should he repair the car or replace it?

Preferences and Risk

In order to develop a consumer (or decision) theory under uncertainty we postulate that individuals have well defined preferences $\succcurlyeq$ over the set of all possible lotteries satisfying the usual properties:

A.1. *Completeness*. $\forall\ l, l'$: $l \succcurlyeq l'$, or $l \succcurlyeq l'$, or both.

A.2. *Transitivity*. $\forall l, l', l''$: $l \succcurlyeq l'$ *and* $l' \succcurlyeq l'' \Rightarrow l \succcurlyeq l''$.

A.3. *Monotonicity*. $\forall l=(x,p), l' = (x',p')$: $\{x > x', p = p'\} \Rightarrow l \succ l'$.

A.4. *Continuity*. $\{l_n \succcurlyeq l'\ \forall n,\ \lim_{n\to\infty} l_n = l\} \Rightarrow l \succcurlyeq l'$.

Preferences and Risk

Examples: Let $l = (x,p)$, $l' = (x',p')$.

[1] Preferences EMV:

$$l \succcurlyeq_{EMV} l' \text{ if } E[l] \geq E[l'].$$

(Comment on St. Petersburg Paradox.)

[2] Preferences maxmin :

$$l \succcurlyeq_{Mm} l' \text{ if } min\ \{x_1,\ldots,x_n\} \geq min\ \{x'_1,\ldots,x'_{n'}\}.$$

[3] Preferences α:

$$l \succcurlyeq_{\alpha} l' \text{ if } E[l^{\alpha}] = \sum_i p_i\, (x_i)^{\alpha} \geq E(l^{\alpha}{}') = \sum_i p_i\, (x'_i)^{\alpha}.$$

Preferences and Risk

Example 2. Order the lotteries: $l = ((4,1),(1/2,1/2))$ and $l' = ((0,5),(1/2,1/2))$, according to the preferences described in [1], [2] and [3] above.

[1] We have

$$E[l] = 1/2\ (4) + 1/2\ (1) = 2{,}5$$

and

$$E\ [l'] = 1/2\ (0) + 1/2\ (5) = 2{,}5.$$

Therefore

$$l \sim_{EMV} l'.$$

Preferences and Risk

Example 2. Order the lotteries: $l = ((4,1),(1/2,1/2))$ and $l' = ((0,5),(1/2,1/2))$, according to the preferences described in [1], [2] and [3] above.

[2] We have

$$min\ \{4,\ 1\} = 1 \text{ y } min\ \{0,\ 5\} = 0.$$

Therefore

$$l \succ_{Mm} l'.$$

Preferences and Risk

Example 2. Order the lotteries: $l = ((4,1),(1/2,1/2))$ and $l' = ((0,5),(1/2,1/2))$, according to the preferences described in [1], [2] and [3] above.

$u(x) = \sqrt{x}$

[3] Assume that α = 0,5. (Note $x^{0,5} = \sqrt{x}$). We have

$$E[l^{0,5}] = 1/2\,\sqrt{4} + 1/2\,\sqrt{1} = 3/2$$

and

$$E[(l')^{0,5}] = 1/2\sqrt{0} + 1/2\sqrt{5} = \sqrt{5}/2 < 3/2.$$

Therefore,

$$l \succ_{0,5} l'.$$

Preferences and Risk

Example 2. Order the lotteries: $l = ((4,1),(1/2,1/2))$ and $l' = ((0,5),(1/2,1/2))$, according to the preferences described in [1], [2] and [3] above.

[3a] Assume $\alpha = 2$. We have

$$E[l^2) = 1/2\ (4^2) + 1/2\ (1^2) = 17/2$$

and

$$E[l'^2] = 1/2\ (0^2) + 1/2\ (5^2) = 25/2.$$

Therefore,

$$l' \succ_2 l.$$

Expected Utility

The preference relations $\succcurlyeq_\alpha$ can represented by a utility function whose value over a lottery

$$l = (x_1, \ldots, x_n, p_1, \ldots, p_n),$$

is the *mathematical expectation* of the random variable

$$l^u = (u(x_1), \ldots, u(x_n), p_1, \ldots, p_n)$$

whose values are the payoffs of the lottery *l, transformed* by the function $u(x) = x^\alpha$.

It seems natural to view u as a *utility* function over payoffs.

Expected Utility

For every function $u: \Re \rightarrow \Re$ we can construct a utility function over lotteries by defining for all $l = (x,p)$

$$v(l) = E[\overset{\overset{Eu(l)}{\shortparallel}}{u}(l)] = \sum_i p_i \, u(x_i).$$

We refer to the function u as a *Bernoulli utility function*, and to the functions over lotteries v with this form (that is, to functions that are a composition of the mathematical expectation and a Bernoulli utility function) as *von Neumann-Morgensten utility functions*.

We simplify notation by writing $Eu(l)$ for $E[u(l)]$.

Expected Utility

Which preferences over lotteries can be represented by von Neumann-Morgensten utility function? To answer this question we need to introduce a new *axiom*.

For $l=(x;p)$, $l' = (y;q)$, $\lambda \in [0,1]$, define the lottery

$$[\lambda l + (1-\lambda)\, l'] = ((x,y),(\lambda p,(1-\lambda)q)).$$

Independence Axiom: $\forall l, l', l'', \lambda \in [0,1]$:

$$l' \succcurlyeq l'' \Rightarrow [\lambda l + (1-\lambda)\, l'] \succcurlyeq [\lambda l + (1-\lambda)\, l''].$$

In which scenarios the Independence Axion does not hold?

Expected Utility

Expected Utility Theorem.

If a preference relation $\succcurlyeq$ over lotteries satisfies axioms A.1, A.2, A.4 and the Independence Axion, then there is a Bernoulli utility function $u\colon \Re \rightarrow \Re$ such that $\forall l, l'$:

$$l \succcurlyeq l' \Leftrightarrow Eu(l) \geq Eu(l').$$

Moreover, if $\succcurlyeq$ satisfies A.3, then the function u is increasing.

EXAMPLE

$$\ell = \left(1, 2, 4 \,;\, \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{3}\right), \quad \ell' = \left(1, 2, 8 \,;\, \tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

HENCE:

$$E[\ell] = \frac{1}{2}(1) + \frac{1}{6}(2) + \frac{1}{3}(4) = \frac{13}{6},$$

$$E[\ell'] = \frac{2}{3}(1) + \frac{1}{6}(2) + \frac{1}{6}(8) = \frac{14}{6}.$$

ASSUME THAT AN INDIVIDUAL'S PREFERENCES FOR LOTTERIES ARE REPRESENTED BY THE BERNOULLI UTILITY FUNCTION

$$u(x) = \ln x.$$

WHICH LOTTERY DOES HE PREFER?

$$E_u(\ell) = \frac{1}{2}\ln 1 + \frac{1}{6}\ln 2 + \frac{1}{3}\ln 4 = \frac{5}{6}\ln 2$$

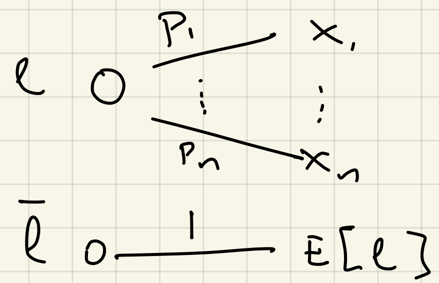
$$E_u(\ell') = \frac{2}{3}\ln 1 + \frac{1}{6}\ln 2 + \frac{1}{6}\ln 8 = \frac{2}{3}\ln 2$$

$$\Longrightarrow \quad \boxed{\ell \succ \ell'}$$

Risk Attitudes

$\ell = (x_1, \ldots, x_n ; p_1, \ldots, p_n)$, $\quad p_i \in (0,1)$, $\quad n \geq 2$. (Non-degenerate)

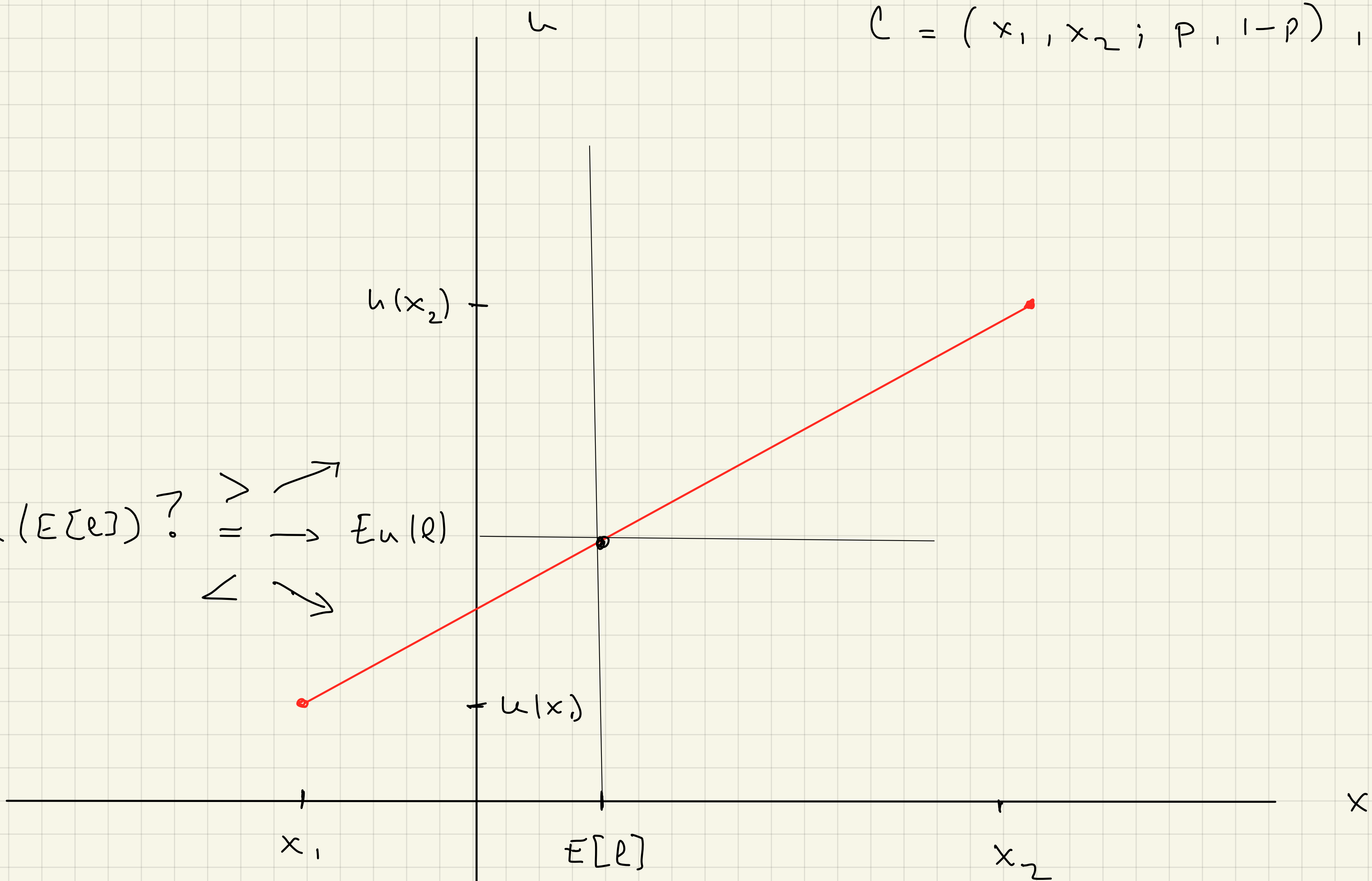
$E[\ell] = \sum_{i=1}^{n} p_i x_i$. $\qquad \bar{\ell} = (E[\ell] ; 1)$



$E[u(\ell)] := Eu(\ell) = \sum_{i=1}^{n} p_i u(x_i)$.

Attitude	$\succsim$	$Eu(\ell) \lesseqqgtr u(E[\ell])$	u	$u \in C^2$
Neutral	$\ell \sim_N \bar{\ell}$	$Eu(\ell) = u(E[\ell])$	Lineal	$u'' = 0$
Averse	$\ell \prec_A \bar{\ell}$	$Eu(\ell) < u(E[\ell])$	Concave	$u'' < 0$
Loving	$\ell \succ_L \bar{\ell}$	$Eu(\ell) > u(E[\ell])$	Convex	$u'' > 0$

$$\ell = (x_1, x_2; p, 1-p), \quad p \in (0,1)$$

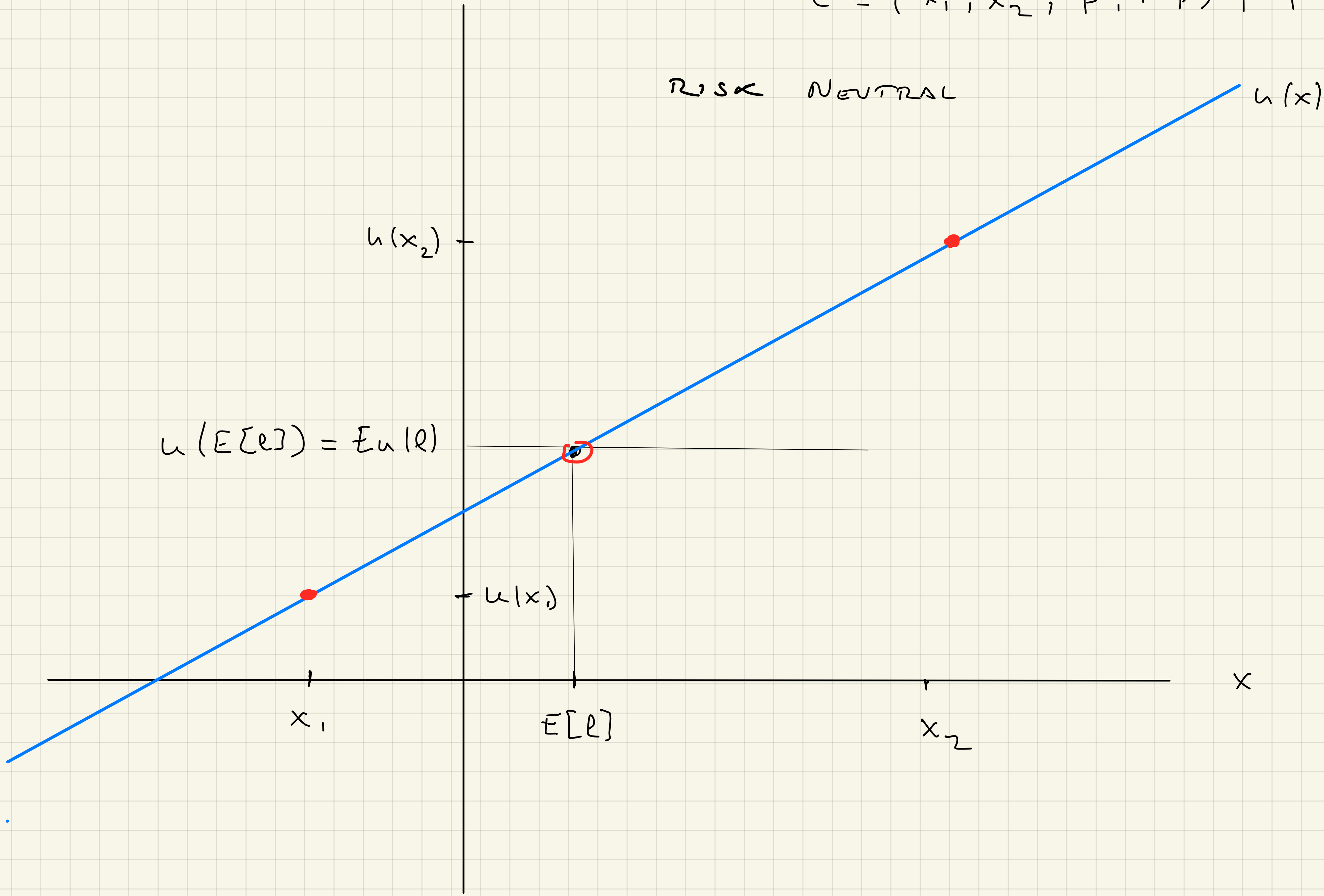


$$u(E[\ell]) \ ? \ \begin{matrix} > \\ = \\ < \end{matrix} \ Eu(\ell)$$

$$E[\ell] = p x_1 + (1-p) x_2$$

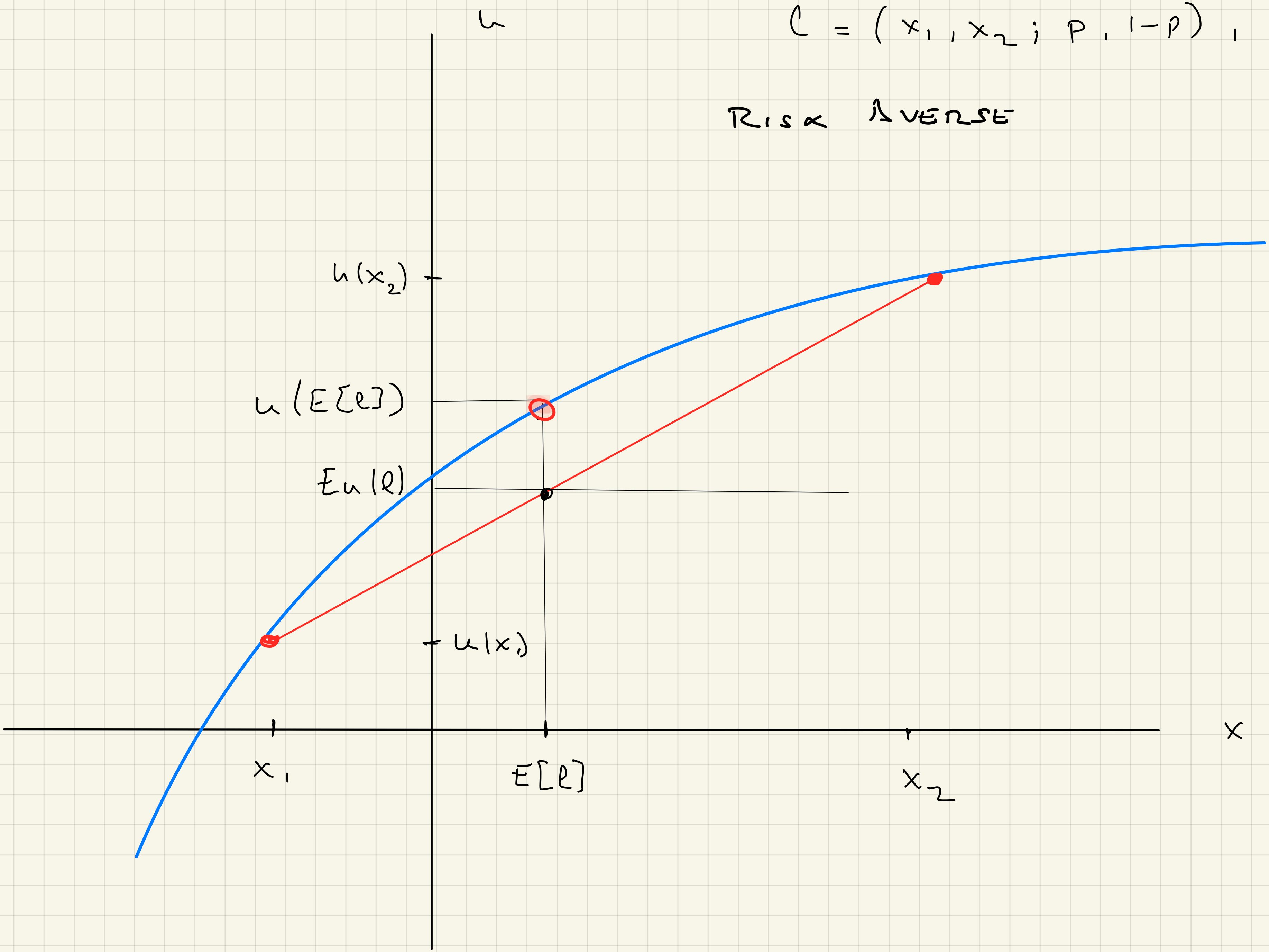
$$Eu(\ell) = p\, u(x_1) + (1-p)\, u(x_2)$$

$\mathcal{L} = (x_1, x_2; p, 1-p), \quad p \in (0,1)$



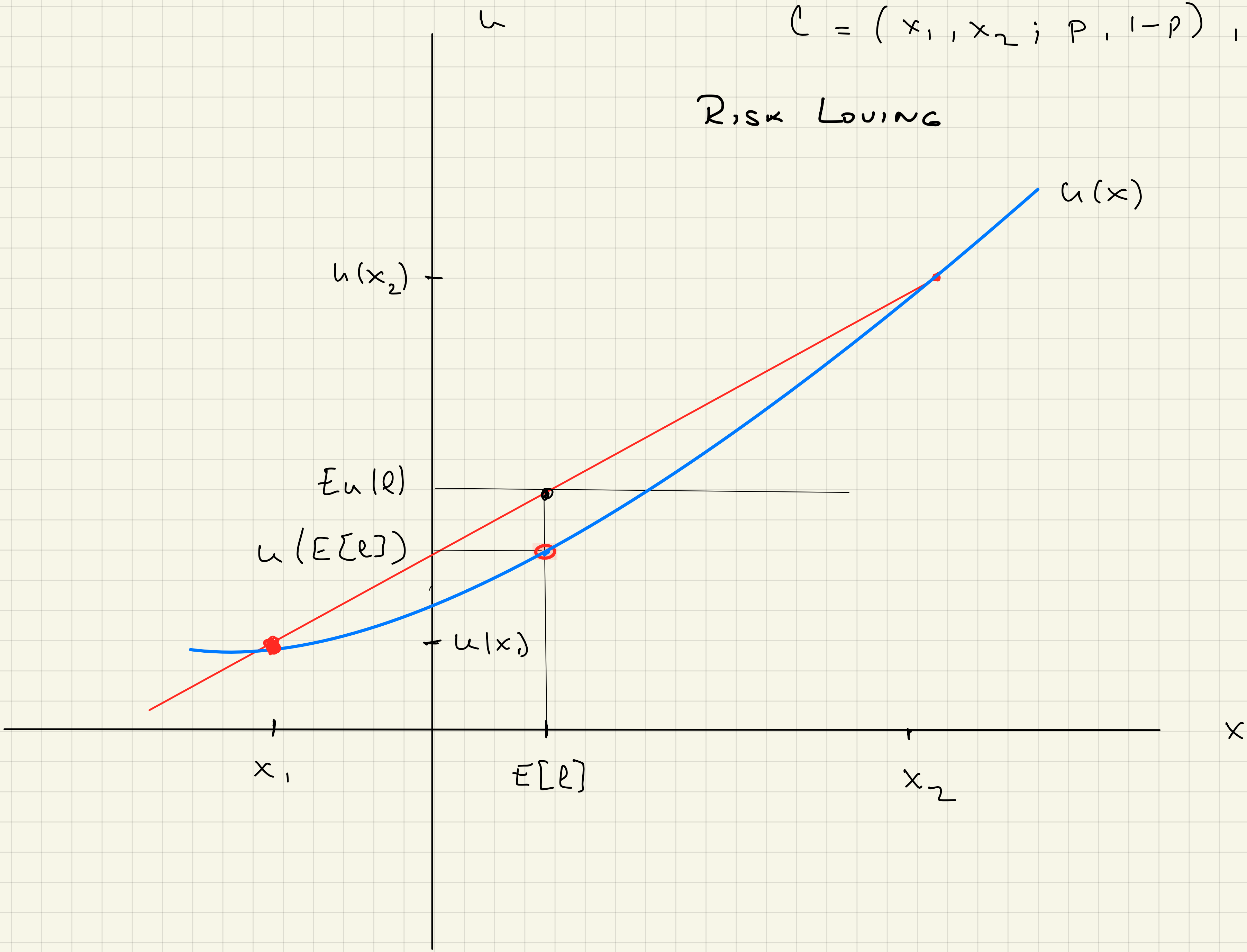
$$L = (x_1, x_2; p, 1-p), \quad p \in (0,1)$$

Risk Averse



$\ell = (x_1, x_2; p, 1-p), \quad p \in (0,1)$

Risk Loving



Risk Attitudes

The risk involved in lotteries is one of the most important factors to order alternatives lotteries and to identify the best alternative.

Obviously, the degree of risk aversion (or the degree of risk attraction) is different for different individuals.

To begin, we propose specific definitions of the concepts of *risk aversion*, *risk neutrality, and risk attraction*.

Risk Attitudes

Let us discuss the consequences of an individual's risk attitude in a simple problem:

Assume that an individual must decide whether to accepts a bet where one can win or lose 10 euros with the same probability, against the alternative of not betting.

We can represent the two alternative lotteries as

$l = (10, -10; 1/2, 1/2)$ and $l' = (0; 1)$.

Risk Attitudes

Since

$$E[l] = E[l'] = 0,$$

it seems natural to postulate that a *risk neutral* individual (that is, someone who feels neither attraction nor aversion to risk) should be indifferent to either lottery; i.e., he would indifferent between betting or not betting. Thus, if $\succcurlyeq_N$ represents his preferences, then

$$l \sim_N l'.$$

Risk Attitudes

An individual who feels attraction to risk should find it exciting betting (lottery *l*), rather than not betting (l'). That is, if $\succcurlyeq_{RL}$ are his preferences of a *risk loving* individual, then

$$l \succ_{RL} l'.$$

Risk Attitudes

And if the individual is a *risk averse*, then he would rather not bet (l') than betting (l). Thus, if $\succcurlyeq_{RA}$ are the preferences of a risk averse individual, then

$$l' \succ_{RA} l.$$

Risk Attitudes

This simple example motivates the following definitions:

We say that a lottery l is *non-degenerate* if it involves at least two different payoffs with positive probability.

In the example we just described, lottery l is non-degenerate, whereas l' is a degenerate lottery.

Risk Attitudes

Let l be a non-degenerate lottery, and let l_c be a (degenerate) lottery that pays $E(l)$ with certainty; that is, $l_c = (E(l); 1)$.

We say that the individual with preferences $\succcurlyeq$ is:

Risk Neutral: if $l \sim l_c$.

Risk Averse: if $l_c \succ l$.

Risk Loving: if $l \succ l_c$.

Risk Attitudes

Exercise. An individual is a participant is a trivial TV program. If she responds correctly to a question, then she has the chance to bet on a second question, and if she respond correctly she can bet on a third question. The payoff to answer correctly the first question is 16 thousand euros, and each time she respond correctly the payoff doubles. However, is she responds incorrectly to a questions, then she loses her entire earnings.

After answering correctly to the first question, a individual who beliefs that she knows the answer to any question with probability 1/2, must decide whether to bet on a second and on a third question.

Represent the problem by means of a decision tree, and solve it assuming that the individual is risk averse. Solve it also assuming that the individual is risk loving, and assuming she is risk neutral.

Risk Attitudes

Proposition 1. An individual's preferences over the lotteries are represented by a Bernoulli utility function u. Let l be a non-degenerate lottery. Then the individual is

- Risk Neutral: if $Eu(l) = u(E[l])$
- Risk Averse: if $Eu(l) < u(E[l])$
- Risk Loving: if $Eu(l) > u(E[l])$.

Risk Attitudes

This simple proposition suggests that there is a relation between the risk attitude of an individual and the curvature of any Bernoulli utility function that represents her preferences.

Let $l = (x_1, x_2; \lambda, 1\text{-}\lambda)$ be a lottery such that $x_1 \neq x_2$ and $0 < \lambda < 1$. We have

$$Eu(l) = \lambda\, u(x_1) + (1\text{-}\lambda)\, u(x_2),$$

and

$$E[l] = \lambda x_1 + (1\text{-}\lambda)x_2.$$

Risk Attitudes

If the individual is risk neutral, then

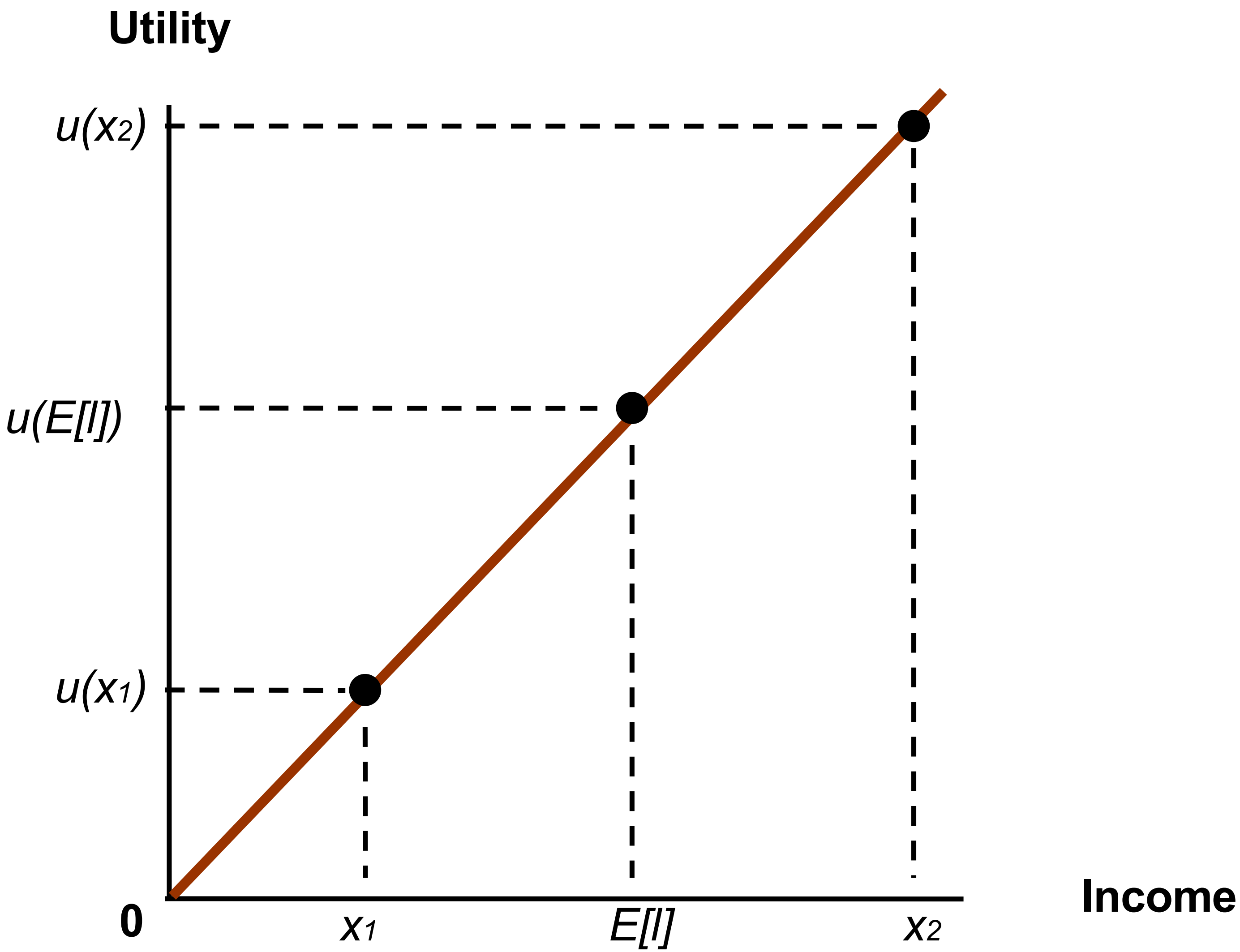
$$\lambda u(x_1) + (1-\lambda)u(x_2) = Eu(l) = u(E(l)) = u(\lambda x_1 + (1-\lambda)x_2).$$

Since x_1, x_2 y λ are arbitrary this implies that u is an afin function; that is,

$$u(x) = a + bx.$$

Note that A.3 implies that $u'(x) = b > 0$.

Risk Neutral



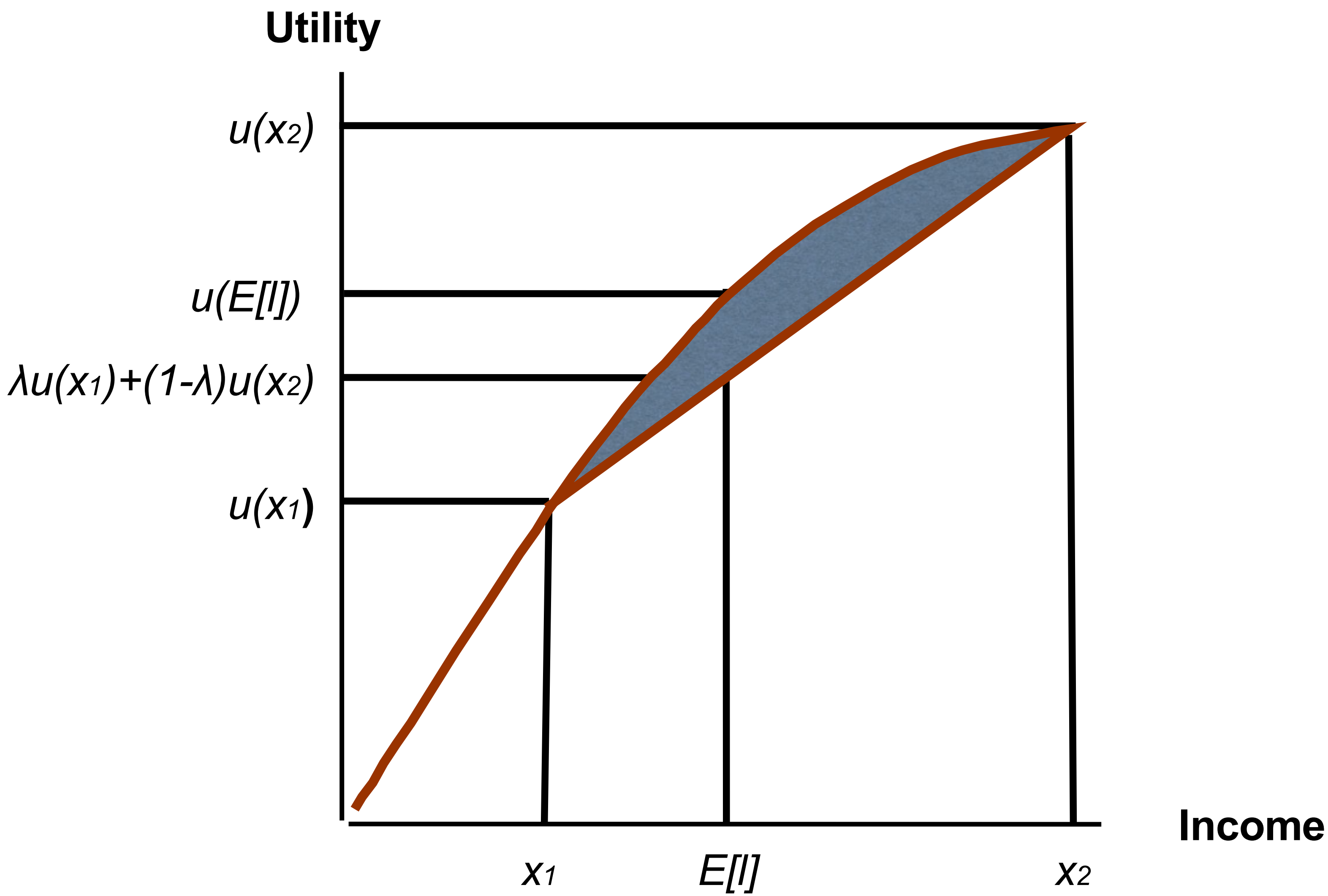
Risk Attitudes

On the hand, since the lottery *l* is non-degenerate, if the individual is risk averse, then

$$\lambda u(x_1) + (1-\lambda)u(x_2) = Eu(l) < u(E[l]) = u(\lambda x_1 + (1-\lambda)x_2).$$

That is, *u* is a (strictly) concave function.

Risk Averse



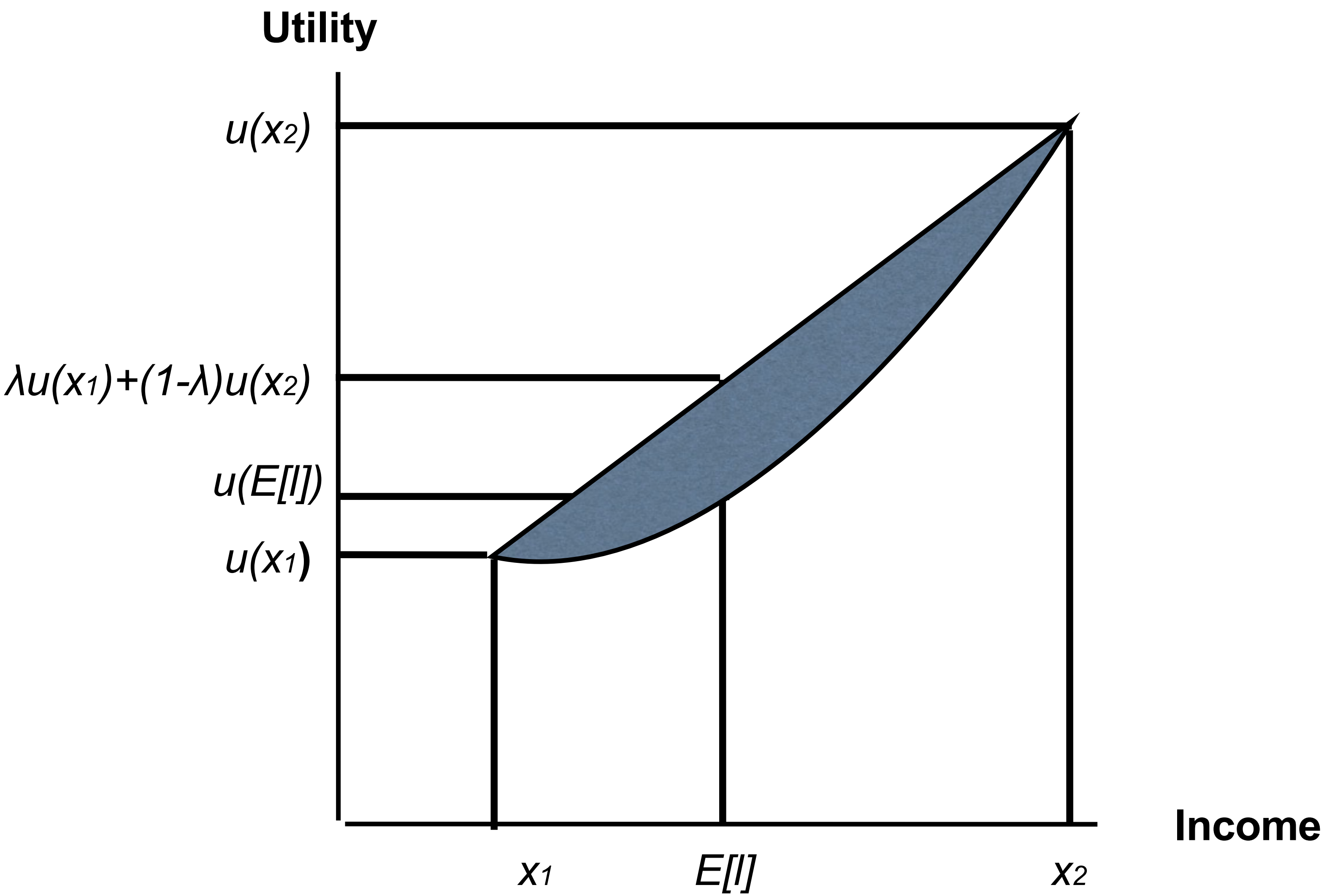
Risk Attitudes

And if the individual is risk loving, then

$$\lambda u(x_1) + (1-\lambda)u(x_2) = Eu(l) > u(E[l]) = u(\lambda x_1 + (1-\lambda)x_2).$$

That is, u is a (strictly) convex function.

Risk Loving



Risk Attitudes

Proposition 2. An individual's preferences over lotteries are represented by a Bernoulli utility function u. Then the individual is

- Risk Neutral: if u is an afin function.
- Risk Averse: if u is a (strictly) concave function.
- Risk Loving: if u is a (strictly) convex function.

Risk Attitudes

If a Bernoulli utility function u is twice differentiable, then the properties of Proposition 2 are easily check: in this case the individual is:

- Risk Neutral: if $u'' = 0$
- Risk Averse: if $u'' < 0$
- Risk Loving: if $u'' > 0$.

(*)

Risk Attitudes

Note that while for each increasing function $f: \Re \rightarrow \Re$, the utility functions of over lotteries v and w such that

$$w(l) = f(v(l))$$

represent the same preferences, this is not the case for Bernoulli utility functions. For example, the Bernoulli utility functions

$$u_1(x) = x, \text{ and } u_2(x) = x^2 = (u_1(x))^2$$

do not represent the same preferences, despite the fact that u_2 is an increasing transformation of u_1.

While u_1 represents the preferences of a risk neutral individual ($Eu_1(l) = E(l)$), the preferences represented by $u_2(x)$ are those of a risk loving individual.

Risk Attitudes

However, if u_2 is an afin transformation of u_1, that is

$$u_2(x) = a + b\, u_1(x),$$

where $b > 0$, then the Bernoulli utility functions u_1 y u_2 represent the same preferences.

The mathematical expectation is a linear operation, that is, for each random variable X and $a,b \in \Re$ we have $E(a+bX) = a + b\, E(X)$. Thus, for every lottery we have

$$Eu_2(l) = a + b\, Eu_1(l).$$

Therefore $\forall l,l'$:

$$Eu_2(l) \geq Eu_2(l') \Leftrightarrow a + b\, Eu_1(l) \geq a + b\, Eu_1(l') \Leftrightarrow Eu_1(l) \geq Eu_1(l').$$

(In particular, all the increasing afin functions represent the same preferences as the Bernoulli utility function $u(x) = x$.)

Risk Attitudes

In order to obtain our last characterization of risk attitudes, we need to introduce the concepts of *certainty equivalent* and *risk premium* of a lottery.

The *certainty equivalent* of lottery l, $CE(l)$, is the solution to the equation

$$u(x) = Eu(l).$$

The *risk premium* of lottery l, $RP(l)$, is

$$RP(l) = E[l] - CE(l).$$

Risk Attitudes

Proposition 3. Let l be any non-degenerate lottery, and let $CE(l)$ be an individual's certainly equivalent. The individual is

- Risk Neutral: if $CE(l) = E[l]$. ✓
- Risk Averse: if $CE(l) < E[l]$. ✓
- Risk Loving: if $CE(l) > E[l]$. ✓

Risk Attitudes

Proposition 4: Let *l* be any non-degenerate lottery, and let *RP(l)* be an individual's certainly equivalent. The individual is

- Risk Neutral: if $RP(l) = 0$.
- Risk Averse: if $RP(l) > 0$.
- Risk Loving: if $RP(l) < 0$.

The Value of Information

In situations of uncertainty the acquisition of new information may allow an individual to increase her welfare by allowing her to select the best alternative depending on the information received.

When acquiring new information is costly, determining whether an individual must incur the cost requires a cost-benefit analysis.

We discuss this in the context of an example.

The Value of Information

Example 1. Jorge has a car that needs to be repaired. He must decide whether to repair it or to replace it with another used car whose price is 1.000 euros. The cost of repairing his current car is uncertain: it may cost either 300 euros with probability 1/3, and 1.200 euros with probability 2/3.

How much will be willing to pay Jorge in order to know the cost of repairing his car?

The Value of Information

Recall that Jorge was risk neutral, so that his preferences are represented by the Bernoulli utility function $u(x) = x$, and that his optimal decision l^* was to repair his car. The expected utility of l^* is

$$Eu[l^*] = E[l^*] = 1/3\ (-300) + 2/3\ (-1200) = -900.$$

The Value of Information

If Jorge knows with certainty the cost of the car repair, then he may condition his decision (whether to replace the car or to replace with the used one he has been offered) on the information received.

Obviously, with perfect information about the repair cost, Jorge would repair if the cost is 300 euros, and he would replace the car otherwise, incurring a cost of 1000 euros

Hence the expected utility of the lottery l_I he faces with perfect information is

$$Eu[l_I] = E[l_I] = 1/3\ (-300) + 2/3\ (-1000) = -766{,}6.$$

The Value of Information

How much is Jorge willing to pay for this information?

If he pays M euros, then his expected utility is

$$Eu(l_I(M)) = 1/3\ (-300-M) + 2/3\ (-1000-M)$$

$$= -(766{,}6 + M).$$

The Value of Information

The maximum quantity Jorge would pay for the information, M^*, is such that the expected utility of the lottery l_I, having paid the information cost M^* is equal to the his expected utility without information, $Eu(l^*)$.

Therefore M^* is the solution to the equation

$$Eu(l_I(M)) = Eu(l^*).$$

The Value of Information

Since

$$Eu(l^*) = -900,$$

The maximum quantity Jorge would pay for the information is the solution to the equation

$$-(766{,}6 + M^*) = -900;$$

that is, $M^* = 133{,}3$ euros.

The Value of Information

We refer to M^* as the *value of (perfect) information*.

Note that the calculation of M^* involves the Jorge's preferences.

The value of information is therefore *subjective.*

That is, there is *objective* of information as its use and impact on the decision problem facing an agent depends on the agent's characteristics.

The Value of Information

Also note that since Jorge is risk neutral and we represent his preferences by $u(x) = x$, then

$$Eu(l) = E[l],$$

and

$$Eu(l_I(M)) = E[l_I] - M.$$

Hence

$$Eu(l_I(M^*)) = Eu(l^*) \Leftrightarrow E[l_I] = E[l^*] - M^*.$$

That is,

$$M^* = E[l_I] - E[l^*].$$

The Value of Information

However, this formula is not correct when the individual is not risk neutral. This is easy to see.

Assume that Jorge's preferences are represented by the Bernoulli utility function

$$u(x) = (1200 + x)^{1/2}.$$

Since $u''(x) < 0$, Jorge is now risk averse.

His expected utility if he repairs the car is now

$$Eu(l_R) = 1/3\ (900)^{1/2} + 2/3\ (0)^{1/2} \approx 10.$$

The Value of Information

However, his expected utility if he replaces the car is

$$Eu(l_S) = (200)^{1/2} \approx 14{,}14.$$

Therefore

$$Eu(l_S) \;\; > Eu(l_R);$$

that is, with this preferences (and beliefs) the optimal decision is to replace the car (rather than to repair it); i.e.,

$$l^* = l_S.$$

The Value of Information

On the other hand, if Jorge knows the repair cost, then he repairs when the cost is 300 euros and replaces the car when it is 1200 euros.

Hence with perfect information his expected utility function is

$$Eu(l_I(M)) = 1/3\ (1200\text{-}300\text{-}M)^{1/2} + 2/3\ (1200\text{-}1000\text{-}M)^{1/2}.$$

The Value of Information

The value of perfect information is now the solution to the equation

$$Eu(l_I(M)) = Eu(l^*);$$

that is,

$$1/3\ (900\text{-}M)^{1/2} + 2/3\ (200\text{-}M)^{1/2} = (200)^{1/2}.$$

Solving we obtain

$$M^* \approx 144{,}23 \neq 133{,}3.$$

The Value of Information

The formula obtained in this example to calculate the value of information,

$$Eu(l_I(M)) - Eu(l^*),$$

applies in general whether information is perfect or imperfect or partial.

Nevertheless, when the information is partial determining the optimal decisions and calculating the expected utility of the corresponding lottery, $l_I(M)$, may be a difficult task.

The Value of Information

The Value of Imperfect Information

Exercise 4. How much will be willing to pay Pedro Banderas to know whether the movie he is considering producing will be played in cinemas?