1. There is only one perishable commodity available for consumption today and tomorrow, and two consumers, Ann and Bob, whose preferences for alternative consumption streams today and tomorrow, $(x, y) \in \mathbb{R}^2_+$, are described by the utility functions $u^A(x, y) = \min\{x, y\}$, and $u^B(x, y) = xy$, respectively.

(a) Assume that the endowment streams are (0,30) for Ann and (20,0) for Bob. Ann and Bob participate in a competitive credit market in which they are the only participants. Calculate the competitive equilibrium rate of interest and consumption streams, and determine whether this allocation is Pareto optimal.

Let us normalize spot prices: $\hat{P}_x = \hat{P}_y = 1$.

Ann's demand of consumption today:

$$\left.\begin{array}{l} \left.\begin{array}{l} x = b \\ y = 30 - (1+r)b \end{array}\right\} \text{B.C.} \\ \left. x = y \right| \text{ OPTIMALITY} \end{array}\right\} \Rightarrow \boxed{b^A(r) = \frac{30}{2+r}}$$

B.C:

CONSOLIDATING THE B.C: $(1+r)x + y = 30 \iff x + \frac{y}{1+r} = \frac{30}{1+r}$

Bob's demand of consumption today :

$$\left.\begin{array}{l} x = 20 + b \\ y = -(1+r)\, b \end{array}\right\} \quad \text{B.C.} \iff (1+r)\,x + y = 20\,(1+r)$$

$$\frac{y}{x} = (1+r) \quad \Big\} \text{ OPTIMALITY}$$

$$\boxed{b^{B}(r) = -10}$$

ALTERNATIVELY:

$$\boxed{\max_{b \in \mathbb{R}} \ (20 + b)\left(-(1+r)\, b\right) \qquad \underline{\text{FOC}} : \ 20 + 2b = 0}$$

<u>CREDIT MARKET CLEARING</u>

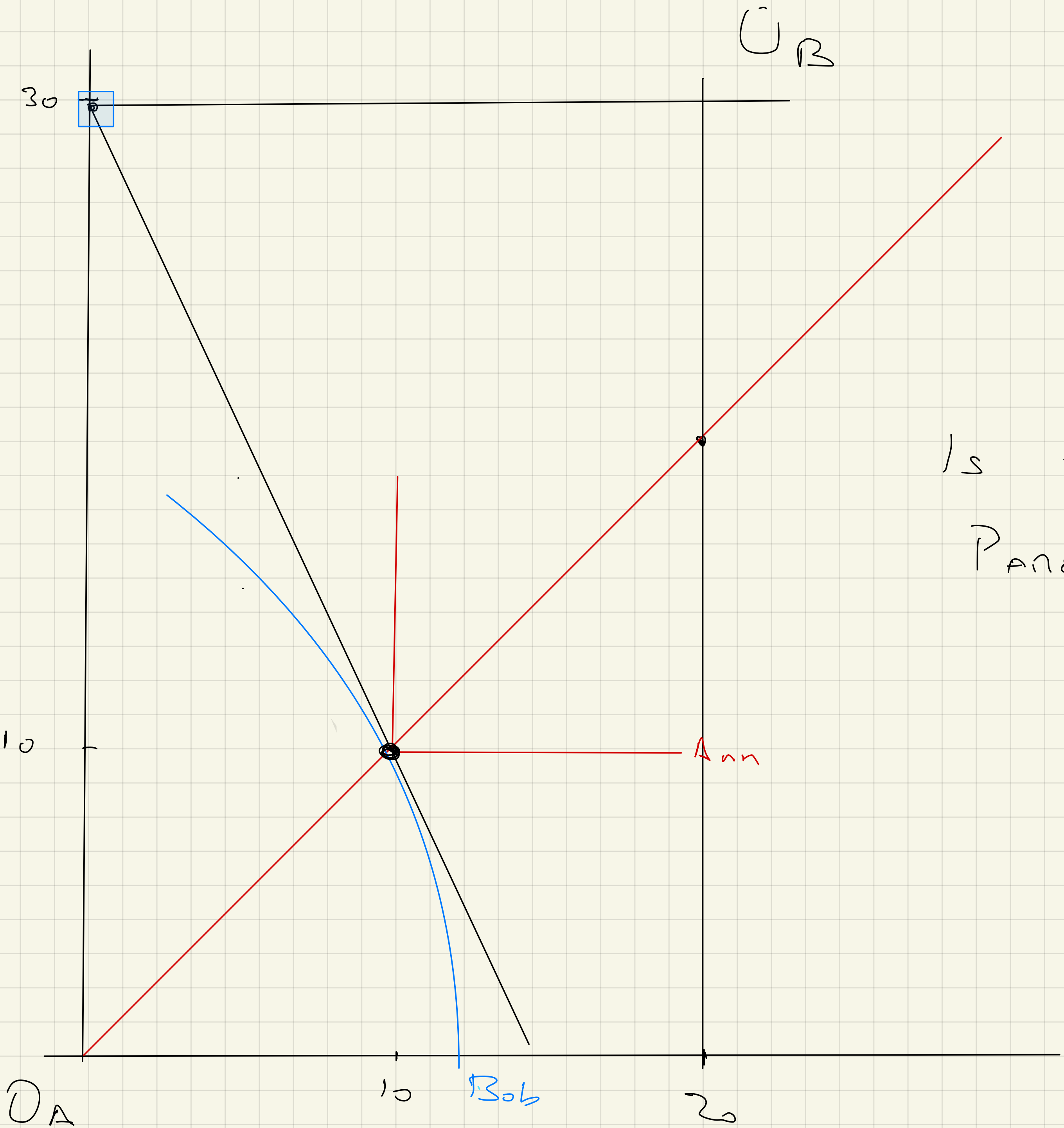
$$b^{A}(r) + b^{B}(r) = 0$$

$$\frac{30}{2+r} - 10 = 0 \iff \underline{\underline{r^{*} = 1}}$$

<u>RADNER</u> CE :

$$(x_A^*, y_A^*) = (10, 10)$$

$$(x_B^*, y_B^*) = (10, 20)$$



Is the Radner CE Pareto optimal?

(b) Calculate the Arrow-Debreu CE of the economy.

Let us normalize spot prices: $p_x = p$, $p_y = 1$.

Ann's demand of consumption today:

BC: $px + y = 30$

OPTIMALITY: $x = y$

$$\boxed{x_A(p) = \frac{30}{p+1}}$$

Bob's demand of consumption today:

BC: $px + y = 20p$

OPTIMALITY: $\frac{y}{x} = p$

$$\boxed{x_B(p) = 10}$$

Market clearing:

$$x_A(p) + x_B(p) = 20$$

$$\frac{30}{p+1} + 10 = 20$$

$p^* = 2.$

Note: $p^* = p_1^* = (1+r^*)\hat{p}_1^*$
$p_2^* = 1 = \hat{p}_2^*$

CE ALLOCATION: $[(10,10),(10,20)].$

(c) Assume that Ann's and Bob's endowments are $(10,0)$ and $(20,0)$, respectively. There is a firm with technology that produces y using x according to the production function $y = 3x$. Ann and Bob have equal shares of the firm.

Calculate the Radner CE of the economy.

FIRM

$$\max_{b \in \mathbb{R}_+} \overbrace{3b - (1+r)b}^{(2-r)b} \qquad b^f(r) = \begin{cases} 0 & \text{IF } 3 < 1+r \\ [0,\infty) & \text{IF } 3 = 1+r \\ \infty \text{ (UNDEFINED)} & \text{IF } 3 > 1+r. \end{cases}$$

HENCE IN A CE $3 \leq 1+r \iff r \geq 2$.

ANN $\quad x_1 = x_2, \quad x_1 = 10+b, \quad x_2 = -(1+r)b \implies b^A(r) = \frac{-10}{2+r}$

BOB $\quad \max_{b \in \mathbb{R}} (20+b)(-\cancel{(1+r)}b) \implies b^B(r) = -\frac{20}{2} = -10$

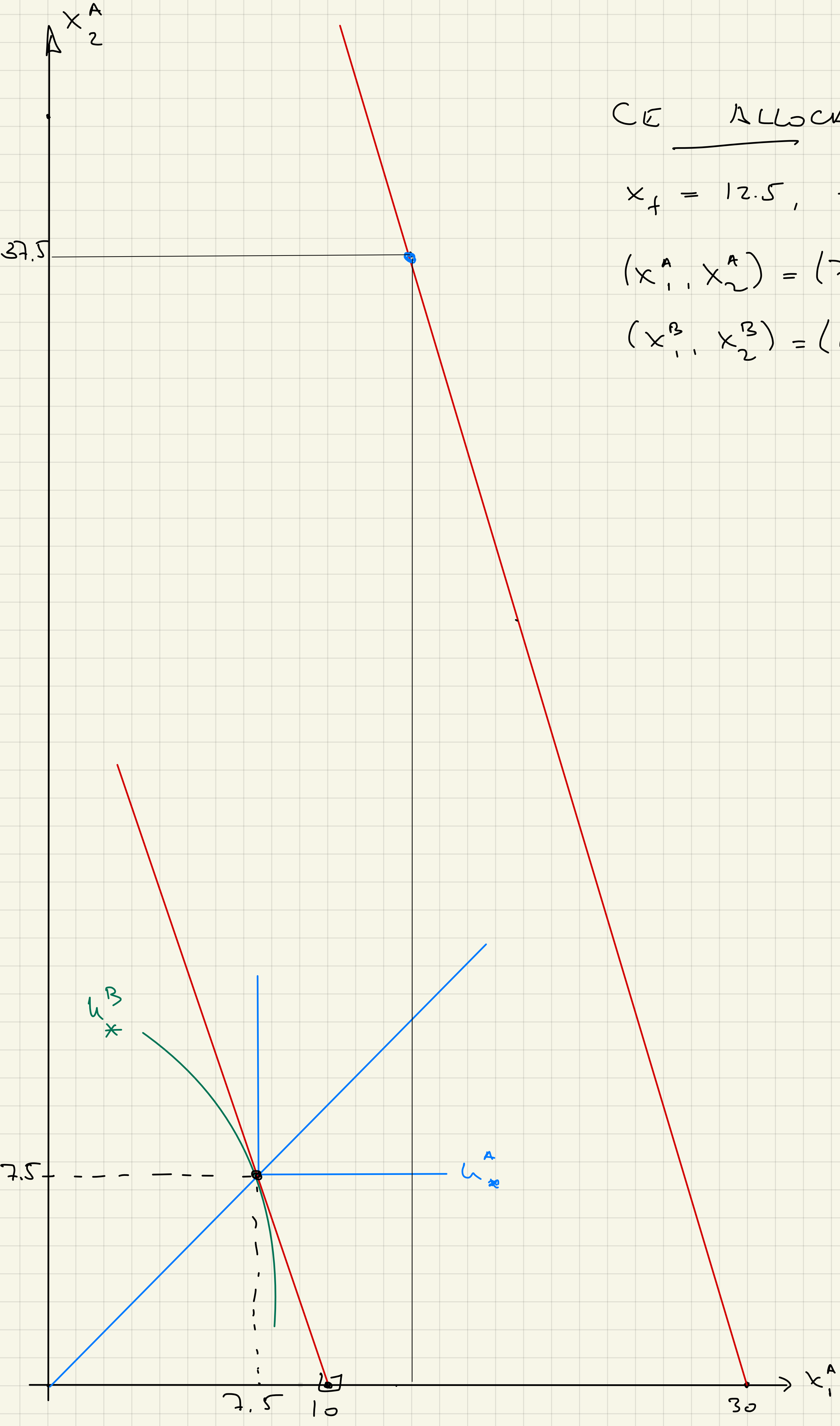
MARKET CLEARING : $B(r) := b^A(r) + b^B(r) + b^f(r) = 0$

For $r < 2$, $B(r) < 0$. But: $b^A(2) = -\frac{10}{4}$, $b^B(2) = -10$, and $-10\left(\frac{5}{4}\right) \in b^f(2)$

HENCE $r^* = 2$.

CE allocation: $x_1^A = 10 + b^A(2) = 7.5 = x_2^A$ $\quad\|\quad$ $x_f = 12.5$

$x_1^B = 10, \quad x_2^B = 30.$ $\quad\quad f(x_f) = 37.5.$



CE ALLOCATION

$x_f = 12.5, \quad f(x_f) = 37.5$

$(x_1^A, x_2^A) = (7.5, 7.5)$

$(x_1^B, x_2^B) = (10, 30)$