Uncertainty: The Role of Financial Assets

Consider a pure exchange economy that operates for two periods. The state of the economy tomorrow is uncertain, and the set of possible tomorrow states of nature is $\Omega = \{\omega_1 \ldots \omega_m\}$

Consider two alternative settings:

A-D: At date $t=1$ there are competitive markets where an agent may trade any physical good today or tomorrow contingent on the state of nature.

R: There are no markets for contingent contracts, but there are spot competitive markets for all goods operating at date 1 and date 2, as well as competitive markets for m securities that operate at date 1.

The returns of each security j are described by a matrix $R_j = (r_{j1}, \ldots, r_{jm})$ where $r_{js} \in \mathbb{R}^{l}_{+}$.

Notation

The Economy : $[(u_1, \bar{x}_1), \ldots, (u_n, \bar{x}_n)]$, where

$$\bar{x}_i \in \mathbb{R}^{l(1+m)}_{+} \qquad u_i : \mathbb{R}^{l(1+m)}_{+} \longrightarrow \mathbb{R}$$

In the setting A-D, a CE is a (p^*, x^*), where $p^* \in \mathbb{R}^{l(1+m)}_{+}$ and $x^* = (x_1^*, \ldots, x_n^*)$ satisfy:

$$\text{(AD.1)} \quad x_i^* \in \underset{x_i \in B_i(p^*)}{\operatorname{argmax}}\, u_i, \quad \forall i \in \{1, \ldots, n\} \text{ and}$$

$$\text{(AD.2)} \quad \sum_{i=1}^{n} x^*_{i1h} = \sum_{i=1}^{n} \bar{x}^*_{i1h} \quad \forall h \in \{1, \ldots, l\}$$

$$\sum_{i=1}^{n} x^*_{i2sh} = \sum_{i=1}^{n} \bar{x}^*_{i2sh}, \quad \forall s \in \{1, \ldots, S\}, \forall h \in \{1, \ldots, l\}.$$

Here

$$B_i(p) = \Big\{ x \in \mathbb{R}^{l(1+m)}_{+} \;\Big|\; \sum_{h=1}^{l} p_{h1}(x_{i1h} - \bar{x}_{i1h}) + \sum_{h=1}^{l} \sum_{s=1}^{m} p_{h2s}(x_{ih2s} - \bar{x}_{ih2s}) \leq 0 \Big\}.$$

In Radner's setting, a CE is a collection $[(\hat{p}^*, q^*), (x_1^*, y_1^*), \ldots, (x_n^*, y_n^*)]$, where $\hat{p}^* \in \mathbb{R}_+^{\ell(1+m)}$, $q^* \in \mathbb{R}_+^m$, satisfying:

$$(R.1) \quad (x_i^*, y_i^*) \in \underset{(x_i, y_i) \in \hat{B}_i(\hat{p}^*, q^*)}{\arg\max} u_i \quad \forall i \in \{1, \ldots, n\},$$

where

$$\hat{B}_i(p, q) = \Big\{ (x_i, y_i) \in \mathbb{R}_+^{\ell(1+m)} \times \mathbb{R}^m \;/$$

$$\sum_{h=1}^{\ell} \hat{p}_{1h}(x_{i1h} - \bar{x}_{i1h}) + \sum_{j=1}^{m} q_j y_{ij} \leq 0, \text{ and}$$

$$\sum_{h=1}^{\ell} p_{2sh}\Big(x_{i2sh} - \bar{x}_{i2sh} - \sum_{j=1}^{m} R_{sh}^j\Big) \leq 0 \quad \forall s \in \{1, \ldots, S\} \Big\}$$

$$(R.2) \quad \sum_{i=1}^{n} x_{i1h}^* = \sum_{i=1}^{n} \bar{x}_{i1h}^* \quad \forall h \in \{1, \ldots, \ell\}$$

$$\sum_{i=1}^{n} x_{i2sh}^* = \sum_{i=1}^{n} \bar{x}_{i2sh}^*, \quad \forall s \in \{1, \ldots, S\}, \forall h \in \{1, \ldots, \ell\}.$$

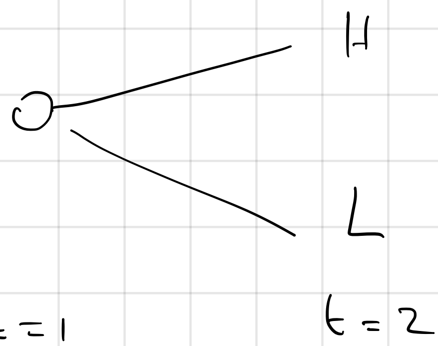
$$(R.3) \quad \sum_{i=1}^{n} y_{ij}^* = 0, \quad \forall j \in \{1, \ldots, m\}.$$

How are A-D and R CE related?

A example (Midterm 2015)

Reformado

A pure exchange economy that operates over two periods. The "state" at $t=2$ may be H or L.



There is a single good (consumption), i.e., $l=1$. There are no contingent markets, but there are spot market each period, and a credit market and a market for a security at date $t=1$. The security pays one unit of the good at date $t=2$ if the state is H, and zero otherwise.

Consumer Budget constraints: Denote by y the units of security the consumer buys, and by $q \geq 0$ the price of the security. Also, normalize the prices of the consumption good to $p_1 = p_L = p_H = 1$. (*)

$\underline{t=1}$ $\qquad x_1 - \bar{x}_1 + qy = b$

$t = 2.H$ $\qquad x_H - \bar{x}_H + (1+r)b - y = 0$

$t = 2.L$ $\qquad x_L - \bar{x}_L + (1+r)b = 0$

Endowments : $[(12, 12, 12), (12, 12, 12)]$

Utilities : $u^A() = \ln x_1 + 2\ln x_H$

$u^B() = \ln x_1 + 2\ln x_L$

Determine the CE.

(*) $\qquad p_1(x_1 - \bar{x}_1) + qy = b$

$$\left.\begin{aligned} p_H(x_H - \bar{x}_H) + (1+r)b - y &= 0 \\ p_L(x_L - \bar{x}_L) + (1+r)b &= 0 \end{aligned}\right\} \quad y = p_H(x_H - \bar{x}_H) - p_L(x_L - \bar{x}_L)$$

THEN $\qquad (x_1 - \bar{x}_1) + \frac{q\,p_H}{p_1}(x_H - \bar{x}_H) + \frac{p_L\left(q - \frac{1}{1+r}\right)}{p_1}(x_L - \bar{x}_L) = 0$

OBSERVATION: ARBITRAGE. Notation: $R = (1+r)$

We can see that in a CE $(1+r)q \geq 1$.

Otherwise, by setting $y = -M$, $b = -qM$ with $M > 0$, we have

$\underline{t=1}$ $\quad x_1 - \bar{x}_1 + \cancel{(-qM)} = \cancel{-qM}$

$t = 2.H$ $\quad x_H - \bar{x}_H = -R(-qM) + (-M)$

$$= (\underbrace{1 - Rq}_{<0})\, M$$

$t = 2.L$ $\quad x_L - \bar{x}_L = RqM$

Here x_L, x_H can be made arbitrarily large, which is incompatible w/ equilibrium.

Governor B: Since he does not care about x_H, he sets

Sets $x_H = 0$ and therefore

$$-12 + Rb - y = 0$$

That is, he sets

$$y = -(12 - Rb)$$

and chooses b to solve:

$$\max_b \; \ell\Big(\underbrace{12 + b + q(12 - Rb)}_{=\,12(1+q) + (1-Rq)b}\Big) + 2\,\ell(12 - Rb)$$

FOC

$$\frac{1 - Rq}{12(1+q) + (1-Rq)b} = \frac{2R}{12 - Rb}$$

i.e.,

$$(1-Rq)(12 - Rb) = 2R\left[12(1+q) + (1-Rq)b\right]$$

i.e.,

$$\underset{4}{\cancel{12}} - \underset{4}{\cancel{12}}Rq - \underset{8}{\cancel{24}}R(1+q) = \cancel{3}R(1-Rq)b$$

$$4 - (8 + 12q)R$$

Solving

$$\boxed{b^B(R,q) = \frac{4 - (8+12q)R}{R(1 - Rq)}}$$

$$b^B\left(\tfrac{1}{2}, 1\right) = \frac{-6}{\frac{1}{4}} = -24$$

Hence

$$y^B(R,q) = -12 + \frac{4 - (8 + 12q)R}{1 - Rq}$$

$$y^B\left(\tfrac{1}{2}, 1\right) = -12 - \frac{6}{\frac{1}{2}} = -24$$

Consumer A

Note: simple, he sets $x_L = 0$, and hence $12 - Rb = 0 \Rightarrow b^A = \frac{12}{R}$

Wants to transfer wealth from $L \to H$. Since $1 - kq \leq 0$ (because $r<0$), the consumer A can increase consumption in H by borrowing and buying security k at date 1. However, he must be able to pay back his debt if the state is L, i.e.,

$$R\, b^A \leq 12$$

Hence

$$\boxed{b^A(R) = \frac{12}{R},} \qquad b^A\left(\tfrac{1}{2}\right) = 24$$

and chooses y to solve

$$\max_{y \in \mathbb{R}} \ \ln\Big(\underbrace{12 + \frac{12}{R}}_{12\left(\frac{1+R}{R}\right)} - qy\Big) + 2\ln\Big(12 \overbrace{- \frac{12R}{R} + y}^{y}\Big)$$

FOC

$$\frac{q}{12\left(\frac{1+R}{R}\right) - qy} = \frac{2}{y} \iff 3qy = 24\,\frac{1+R}{R} \qquad \underline{b^A\left(\tfrac{1}{2}, 1\right) = 24}$$

$$y^A(R,q) = \frac{8(1+R)}{qR}; \quad y^A\left(\tfrac{1}{2}, 1\right) = 24$$

Therefore, the Radner CE of this economy is $(r^*, q) = (-\frac{1}{2}, 1)$ and $x^* = [(12, 24, 0), (12, 0, 24)]$.

Clearly this allocation is P.O.

Is this usually the case?

What if there was only a market for credit?

In this case, there is no activity in the credit market (verify it!), and therefore the CE is the initial allocation, which is not P.O.

The issue is that it is not feasible, with only a credit, to transfer consumption across the two states.

Also, we can easily identify the A-D CE prices of the contingent commodities → do it!

How rich needs to be the set of securities in order for a Radner economy to generate equilibrium allocations that are P.O.?

An **EXAMPLE**: (Need to define the notion of Radner CE for this economy.)

Assume $T=1$, and that there are m securities. Securities $s \in \{1 \dots m\}$ pays 1 unit of good 1 (the numeraire) in state s, and nothing otherwise.

Proposition. (Mention only result (2).)

(1) Let (p, x) be an AD CE such that $p_{1s} > 0$ $\forall s \in \{1 \dots S\}$. Then $(\hat{p}, q, x, y)$, where

$$\hat{p} = p, \quad q_s = p_{1s}, \quad y_{is} = \frac{1}{p_{1s}} \sum_{h=1}^{\ell} (x_{ihs} - \bar{x}_{ihs})$$

is a R CE

(2) Let $(\hat{p}^*, q^*, x^*, y^*)$ be a R CE such that $\hat{p}^*_{1s} > 0$ $\forall s \in \{1 \dots m\}$. Then (p^*, x^*), where $p^*_{hs} = q^*_s \hat{p}^*_{hs} / \hat{p}^*_{1s}$, forms a AD CE.

In this economy, a Radner CE is a collection $(\hat{p}^*, q^*, x^*, y^*)$ such that

(1) $\forall i \in \{1, \dots, n\}$, (x_i^*, y_i^*) solves

$$\max_{(x_i, y_i) \in \hat{B}_i(\hat{p}^*, q^*)} u_i(x_i)$$

where

$$\hat{B}_i(\hat{p}, q) = \{ (x_i, y_i) \in \mathbb{R}_+^{\ell(1+m)} \times \mathbb{R}^m \mid$$

$$\sum_{s=1}^{m} q_s y_{is} = 0$$

$$\forall s \in \{1, \dots, m\}: \sum_{h=1}^{\ell} \hat{p}_{hs}(x_{ihs} - \bar{x}_{ihs}) = \hat{p}_{1s} y_s$$

(2) $\forall s \in \{1, \dots, m\}$: $\sum_{i=1}^{n} y_{is}^* = 0$

$\forall (h, s) \in \{1, \dots, \ell\} \times \{1, \dots, n\}$: $\sum_{i=1}^{n} x_{ihs} = \sum_{i=1}^{n} \bar{x}_{ihs}$

Proof:

~~(I)~~ (I.a) $x_i(p) \in \hat{B}^i_x(\hat{p}, q)$

(I.b) $\hat{B}^i_i(\hat{p}, q) \subset B^i(p)$

$(I.a) + (I.b) \Rightarrow \quad x_i(p) = x_i(\hat{p}, q)$

Market clearing in spot market is implied by market clearing in contingent markets.

As for the security markets, we have

$$\underline{\sum_{i=1}^{n} y_{is}} = \frac{1}{p_{1s}} \sum_{i=1}^{n} \left(\sum_{h=1}^{\ell} p_{hs} \left(x_{ihs} - \bar{x}_{ihs} \right) \right)$$

$$= \frac{1}{p_{1s}} \sum_{h=1}^{\ell} p_{hs} \sum_{i=1}^{n} \left(x_{ihs} - \bar{x}_{ihs} \right) = \underline{0}$$

(II) $B^i(p^*) \subset \hat{B}^i_x(\hat{p}^*, q^*)$

Assume $\tilde{x}_i \in B^i(p)$. Let $y_{is} = \frac{1}{p_{1s}} \sum_{h=1}^{\ell} \hat{p}_{hs} \left(\tilde{x}_{ihs} - \bar{x}_{ihs} \right)$

Then $\quad \sum_{s=1}^{m} q_s y_{is} = \sum_{s=1}^{m} \frac{q_s}{p_{1s}} \left(\sum_{h=1}^{\ell} \hat{p}_{hs} \left(\tilde{x}_{ihs} - \bar{x}_{ihs} \right) \right)$

$$= \sum_{s=1}^{m} \sum_{h=1}^{\ell} p_{hs} \left(\tilde{x}_{ihs} - \bar{x}_{ihs} \right).$$

Then $\tilde{x}_i \in \hat{B}^i_x(\hat{p}, q)$, i.e., $\tilde{x} \in \hat{B}^i_x(\hat{p}, q) \Rightarrow \tilde{x}^i \in B^i(p)$.

Therefore, $x_i \in \operatorname{argmax} \hat{B}^i(p)$. And since spot markets clear, so do contingent markets.

Need to show that $x_i^* \in B_i(p^*)$.

i.e.,

$$\sum_{s=1}^{m} \sum_{h=1}^{\ell} p_{hs}^* \left(x_{ihs}^* - \bar{x}_{ihs} \right) = 0$$

$$= \sum_{s=1}^{m} \sum_{h=1}^{\ell} \frac{q_s^* \hat{p}_{hs}^*}{\hat{p}_{1s}^*} \left(x_{ihs}^* - \bar{x}_{ihs} \right)$$

$$= \sum_{s=1}^{m} \frac{q_s^*}{\hat{p}_{1s}} \sum_{h=1}^{\ell} \hat{p}_{hs}^* \left(x_{ihs}^* - \bar{x}_{ihs} \right)$$

$$= \sum_{s=1}^{m} \cancel{\frac{q_s^*}{\hat{p}_{1s}^*}} \cancel{\hat{p}_{1s}^*} y_s^* = 0$$

— x —

What R's result uncovers is that if the matrix of securities returns R is such that rank $R = S$, then a Radner CE allocation is also an Arrow-Debreu CE allocation.

The condition on the rank assures that an agent is able to transfer monetary income across states of nature as he wishes.

This result generalizes if the set of securities is rich enough to expand the set consumption bundles $\underline{\mathbb{R}}_+^{TSl}$ available when there is a <u>complete</u> market structure.

Note that securities provide a more "<u>economical</u>" market structure:

m	l	ml	$m+l$
2	2	4	4
3	3	6	9
100	100	10,000	200

What if markets are incomplete?

The R CE may be inefficient.

Examples are easy to construct.

(e.g. Ex. 4a – see next pages.)

If markets are missing (due to info asymmetries, transaction costs, whatever), are the CE at least PO relative to the restricted trading opportunities available given the technological or informational constraints?

The answer is NO: contract theory shows us that the existence of a judicial authority that can enforce contracts allows agents to improve upon the market trading opportunities.

In addition, the inefficiency of the outcomes of this CE economies gives room to government intervention.

ILLUSTRATION: EXERCISE 1.4 in the list

(a)

Budget Constraint is

$$X_0 = 10 + b$$

$$X_s = 15 - (1+r)\,b$$

$$X_c = 15 - (1+r)\,b$$

Hence

$$\max_{(\,)\ \text{s.t. } bc} u(x_0, x_s, x_c) \equiv \max_b \; (10+b) + 5\pi_i \ln(15-(1+r)b) + 5(1-\pi_i)\ln(15-(1+r)b)$$

$$\equiv \max_b \; (10+b) + 5\ln[15-(1+r)b]$$

Same for both individuals!

In a CE $b_1(r) = b_2(r) = 0$.

FOC $\quad 1 - \dfrac{5(1+r)}{15-(1+r)b} = 0 \iff b_i(r) = \dfrac{15}{1+r} - 5$

$$b_i(r) = 0 \iff \boxed{r^* = 2}$$

Clearly, this is not PO.

(b) $\bar{w}^i = (10, 15, 15)$

Now budget constraint on $(p_0 = 1)$

$$x_0 = 10 + p_s(15 - x_s) + p_c(15 - x_c)$$

$\rightsquigarrow$

$$\max_{x_s, x_c} \{10 + \quad\} + 5\pi_i \ln x_s + 5(1-\pi_i)\ln x_c$$

FOC

$$p_s = \frac{5\pi_i}{x_s} \qquad \pi_A = \frac{1}{2}$$

$$\pi_B = \frac{3}{4}$$

$$p_c = \frac{5(1-\pi_i)}{x_c}$$

i.e., $$(x_s^A, x_c^A) = \left(\frac{5}{2p_s}, \frac{5}{2p_c}\right)$$

$$(x_s^B, x_c^B) = \left(\frac{15}{4p_s}, \frac{5}{4p_c}\right)$$

$$\frac{10}{4p_s} + \frac{15}{4p_s} = 30 \Rightarrow p_s = \frac{5}{24}$$

$$\frac{10}{4p_c} + \frac{5}{4p_c} = 30 \Rightarrow p_c = \frac{1}{8}$$

$$\boxed{(x_s^A, x_c^A) = (12, 20), \quad (x_s^B, x_c^B) = (18, 10)}$$

It is easy to see that $x_0^A = x_0^B = 10$.

E.g.,

$$x_0^A = 10 + p_s(15 - x_s) + p_c(15 - x_c)$$

$$= 10 + \frac{5}{24}(15 - 12) + \frac{1}{8}(15 - 20)$$

$$= 10.$$

Here, the final allocation is

$$x^* = [(10, 12, 20), (10, 18, 10)],$$

and of course, both consumers are better off: Consumer B transfers consumption from the cloudy state (which she thinks is less likely) to the sunny state (which she thinks is more likely). Consumer A is happy to accept (even though she thinks that both states are equally likely) because the terms of trade are favorable.

(c)

Normalize spot prices: $\hat{p}_0 = \hat{p}_c = \hat{p}_s = 1$.

The Budget Constraints are

(1) $x_0 = 10 + b - 9y$

(2) $x_s = 15 - (1+r)b$

(3) $x_c = 15 - (1+r)b + y$

i.e.,

$$b = \frac{1}{1+r}(15 - x_s) \qquad (2)$$

$$y = x_c - 15 + 15 - x_s \qquad (3)+(2)$$

$$= x_c - x_s$$

Then

$$x_0 = 10 + \frac{15 - x_s}{1+r} - 9(x_c - x_s)$$

i.e.,

$$\boxed{x_0 + \left(\frac{1}{1+r} - 9\right)x_s + 9x_c = 10 + \frac{15}{1+r}}$$

Hence $p_s^* = \frac{1}{1+r} - 9 = \frac{5}{24}$ $\Rightarrow \frac{1}{1+r} = \frac{5}{24} + \frac{1}{8} = \frac{8}{24} = \frac{1}{3}$

$p_c^* = 9^* = \frac{1}{8}$ $\qquad 1+r = 3 \Rightarrow r^* = 2$

$$(x_s^A, x_c^A) = (12, 20) \Rightarrow b^A(r^*, q^*) = \frac{1}{3}(15-12) = 1$$

$$y^A(r^*, q^*) = 20 - 12 = 8$$

$$(x_s^B, x_c^B) = (18, 10) \Rightarrow b^B(r^*, q^*) = \frac{1}{3}(15-18) = -1$$

$$y^A(r^*, q^*) = 10 - 18 = -8$$