Final Exam
(May 28, 2015)

Exercise 1. Consider an economy that extends over two periods, today and tomorrow, and where there are two consumers, A and B, and a single perishable good. The state of nature tomorrow can either be H or L. Consumers' preferences over consumption today and tomorrow are represented by the utility functions $u_A(c_0, c_H, c_L) = c_0 c_H$, and $u_B(c_0, c_H, c_L) = c_0 c_L$. Both A and B are endowed with four units of the good in each of the two periods (regardless of whether the state in the second period is H or L).

(a) (10 points) Suppose that the only market available is a credit market. What will be the competitive equilibrium interest rate r^* and how much will each person borrows $b_i(r^*)$? Is the resulting allocation Pareto optimal?

Solution: Each consumer $i \in \{A, B\}$ solves the problem

$$\max_{[(c_0,c_H,c_L),b]\in\mathbb{R}^3_+\times\mathbb{R}} u_i(c_0, c_H, c_L)$$

$$\begin{array}{llll} s.t. & c_0^i & \leq & 4 + b_i \\ & c_H^i & \leq & 4 - (1+r)b_i \\ & c_L^i & \leq & 4 - (1+r)b_i. \end{array}$$

Solving this problem we get

$$b_i(r) = -\frac{2r}{1+r}.$$

Hence market clearing implies

$$b_A(r) + b_B(r) = 2\left(-\frac{2r}{1+r}\right) = 0,$$

which implies

$$r^* = 0 = b_A(r^*) = b_B(r^*).$$

The resulting allocation is

$$[(c_0^A, c_H^A, c_L^A), (c_0^B, c_H^B, c_L^B)] = [(4, 4, 4), (4, 4, 4)].$$

This allocation is not Pareto optimal because the allocation $[(4, 8, 0), (4, 0, 8)]$ *is also feasible and is Pareto superior.*

(b) (10 points) Now suppose that in addition to a credit market there is a security that pays one unit of consumption tomorrow only if event H occurs. Let q denote the market price of this security (in units of consumption today). Determine the competitive equilibrium interest rate and security price. Is the resulting allocation Pareto optimal? Does equilibrium involve either consumer *selling short*? (Hint: try and consolidate the constraint that are binding and relate the resulting consumer's problem to that she faces in a walrasian economy.)

Solution: Each consumer $i \in \{A, B\}$ solves the problem

$$\max_{[(c_0,c_H,c_L),b,z]\in\mathbb{R}^3_+\times\mathbb{R}^2} u_i(c_0, c_H, c_L)$$

$$\begin{aligned}
s.t. \quad (0)\ c_0^i &\leq 4 - qz_i + b_i \\
(H)\ c_H^i &\leq 4 + z_i - (1+r)b_i \\
(L)\ c_L^i &\leq 4 - (1+r)b_i.
\end{aligned}$$

Obviously the constraint (0) *and* (H) *are binding for consumer* A*, and the constraint* (0) *and* (L) *are binding for consumer* B*. Manipulating these constraints we may write the consumer problem in either case as*

$$\max_{[(c_0,c_H,c_L),b,z]\in\mathbb{R}^3_+\times\mathbb{R}^2} u_i(c_0, c_H, c_L)$$

$$s.t.\ c_0^i + qc_H^i + \left(\frac{1}{1+r} - q\right) c_L^i \leq 4 + 4q + 4\left(\frac{1}{1+r} - q\right).$$

This problem is equivalent to that the consumer faces when there is a complete set of markets. Consumer A*'s demands are* $c_L^A(q,r) = 0$ *and* $c_0^A(q,r)$, $c_H^A(q,r)$ *are obtained by solving*

$$\begin{aligned}
\frac{c_H^A}{c_0^A} &= \frac{1}{q} \\
c_0^A + qc_H^A &= \frac{4\,(2+r)}{1+r}
\end{aligned}$$

Hence the

$$c_0^A(q,r) = \frac{2\,(2+r)}{1+r},\ c_H^A(q,r) = \frac{2\,(2+r)}{q\,(1+r)}$$

Likewise, Consumer B*'s demands are* $c_H^A(q,r) = 0$ *and* $c_0^A(q,r)$, $c_L^A(q,r)$ *are obtained by solving*

$$\begin{aligned}
\frac{c_L^B}{c_0^B} &= \left(\frac{1}{1+r} - q\right)^{-1} \\
c_0^B + \left(\frac{1}{1+r} - q\right) c_L^B &= \frac{4\,(2+r)}{1+r}
\end{aligned}$$

Hence the

$$c_0^B(q,r) = \frac{2\,(2+r)}{1+r},\ c_L^B(q,r) = \frac{2r+4}{1-q(1+r)}$$

Market clearing implies

$$\begin{aligned} c_H^A(q,r) + c_H^B(q,r) &= 8 \\ c_L^A(q,r) + c_L^B(q,r) &= 8 \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{2\,(2+r)}{q\,(1+r)} &= 8 \\ \frac{2r+4}{1-q(1+r)} &= 8, \end{aligned}$$

which implies $q^* = \frac{1}{2}, r^* = 0$. *The resulting allocation is* $[(c_0^A, c_H^A, c_L^A), (c_0^B, c_H^B, c_L^B)] = [(4,8,0),(4,0,8)]$. *This allocation is Pareto optimal.*

Moreover, since

$$c_0^A = 4 - (1+r^*)\, b_A^* = 0,$$

then $b_A^* = -4 = -b_B^*$. *And since*

$$c_H^B = 4 + z_B - (1+r)b_B = 0,$$

then $z_B = -8$, *so that consumer* B *is selling short some units of the security – she has only four units of the good in this state, but sells 8 units knowing that she is also lending and will have enough units to deliver the eight units.*

Exercise 2. Consider the contract design problem of a risk-neutral Principal who wants to hire an Agent. There are two types of agents, H and L, both with the same preferences, which are represented by the Bernuilli utility function $u(x) = x$, and the same reservation utilities, given by $\underline{u} = 1$, but with different costs of effort, $v_i(e) = k_i e$ with $k_H = 1$ and $k_L = 2$. Let q denote the probability that the Agent is of type H. The Principal's expected revenue is a function of the Agent's effort, which is verifiable, and is given by $\bar{x}(e) = 8 \ln e$ for $e \in [1, \infty)$.

(a) (20 points) Calculate the contract the Principal will offer to each type of agent if types were observable. Illustrate your findings providing a graph of the Principal's demand of effort for each type of agent as well as the supply of effort of each type of agent.

Solution: With complete information the optimal contracts $[(e_H^*, w_H^*), (e_L^*, w_L^*)]$ *solve the systems*

$$\begin{aligned} \frac{8}{e_H} &= \frac{1}{1} \\ w_H &= e_H + 1 \ (PC_H) \end{aligned}$$

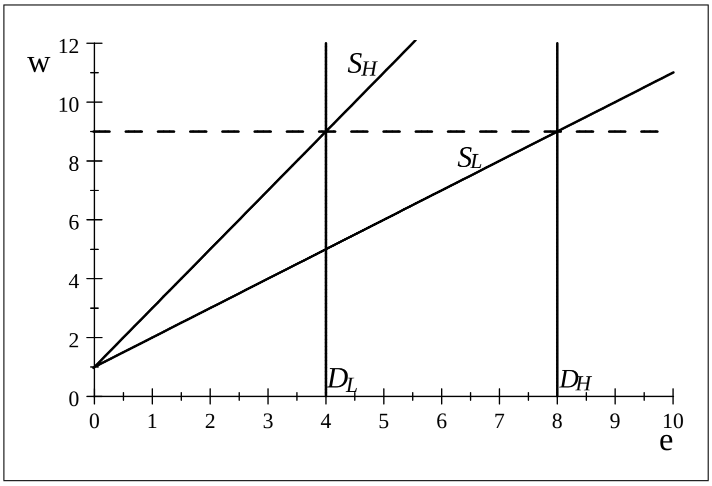
and

$$\begin{aligned} \frac{8}{e_L} &= \frac{2}{1} \\ w_L &= 2e_L + 1 \ (PC_L). \end{aligned}$$

Hence $[(e_H^*, w_H^*), (e_L^*, w_L^*)] = [(8, 9), (4, 9)]$. *Since*

$$\pi(e_H^*, w_H^*) = 8 \ln 8 - 9 > 0, \ \pi(e_L^*, w_L^*) = 8 \ln 4 - 9 > 0$$

This menu is optimal.



(b) (20 points) Identify the optimal menu of contracts the Principal will offer if Agents' types are private information (that is, not observed by the Principal), assuming that the Principal wants to hire the Agent whichever may be her type. Determine for which values of q this menu yields more profits than hiring only the high type worker.

Solution: As seen in class, the system of equations identifying the optimal menu with asymmetric information is:

$$\begin{aligned} \frac{8}{e_H} &= 1 \\ \frac{8}{e_L} &= 2 + \frac{q}{1-q} \\ w_L &= 2e_L + 1 \\ w_H - e_H &= w_L - e_L \end{aligned}$$

Solving the system we get

$$[(\tilde{e}_H, \tilde{w}_H), (\tilde{e}_L, \tilde{w}_L)] = \left[\left(8, \frac{26-17q}{2-q}\right), \left(\frac{8(1-q)}{2-q}, \frac{18-17q}{2-q}\right)\right].$$

The Principal's profit with this menu is

$$\begin{aligned} \tilde{\Pi} &= q\left(8\ln e_H - w_H\right) + (1-q)\left(8\ln e_L - w_L\right) \\ &= q\left(8\ln 8 - \frac{26-17q}{2-q}\right) + (1-q)\left(8\ln\left(\frac{8(1-q)}{2-q}\right) - \frac{18-17q}{2-q}\right) \\ &= 8(1-q)\ln\frac{8(1-q)}{2-q} + 8q\ln 8 - 9. \end{aligned}$$

The profit of the Principal if he offers only the contract acceptable by Agent H, $(e_H^*, w_H^*) = (8, 9)$, *is* $\pi\left(e_H^*, w_H^*\right) = 8\ln 8 - 9$. Hence

$$\begin{aligned} \tilde{\Pi} - \pi\left(e_H^*, w_H^*\right) &= 8(1-q)\left(\ln\frac{8(1-q)}{2-q} - \ln 8\right) \\ &= 8(1-q)\left(\ln\frac{1-q}{2-q}\right) \\ &= 8(1-q)\left(\ln(1-q) - \ln(2-q)\right) \\ &< 0, \end{aligned}$$

and therefore it the menu identify in part (b) is never optimal, and the optimal contract is to offer exclusively the contract (e_H^*, w_H^*), *and hence not to contract Agent L.*

Exercise 3. A competitive market provides insurance to a population of individuals with preferences represented by the Bernuilli utility function $u(x) = \ln x$, where x is the individual's disposable income, who have an initial wealth $W = 5$ and face the risk of a monetary lose $L = 4$. For a fraction $\lambda \in (0, 1)$ of the individuals the probability of losing L is $p^L = 1/4$ whereas for the remaining fraction $1 - \lambda$ this probability is $p^H = 1/2$. This information is common knowledge to all market participants. At the time an individual is signing a policy, an insurance company does not know whether her probability of loosing L is p^L or p^H.

(a) (10 points) Which policies will be offer in a competitive equilibrium? (Assume that a competitive equilibrium exist, and identify the policies offered.) (Show your solution graphically first, and then do the algebra.)

Solution. As seen in class a competitive equilibrium, when it exists, is separating. In a separating equilibrium high risk agents get full insurance, that is $(I_H^*, D_H^*) = (p^H L, 0)$. Hence the expected utility for a high risk individual is

$$u(W - p^H L) = \ln\left(W - p^H L\right) = \ln\left(5 - \left(\frac{1}{2}\right)4\right) = \ln 3$$

Low risk individuals get partial insurance: $(I_L^*, D_L^*) = (p^L(L-D), D)$, where D must leave the high risk individuals indifferent between the policy (I_H^*, D_H^*) and the policy $(p^L(L - D), D)$, that is,

$$p^H u(W - p^L(L - D) - D) + \left(1 - p^H\right) u(W - p^L(L - D)) = u(W - p^H L)$$

Substituting, this equation becomes

$$\frac{1}{2}\ln\left(5 - \frac{1}{4}(4 - D) - D\right) + \frac{1}{2}\ln(5 - \frac{3}{4} + \frac{D}{2}) = \ln 3.$$

which may be written as

$$\left(5 - \frac{1}{4}(4 - D) - D\right)\left(5 - \frac{3}{4} + \frac{D}{2}\right) = 3^2$$

or

$$-\frac{3}{8}D^2 - \frac{19}{16}D + 8 = 0.$$

Hence

$$D^* = -\frac{4}{3}\left(\frac{19}{16} - \sqrt{\left(\frac{19}{16}\right)^2 + 12}\right) \simeq \frac{10}{3}.$$

For these policies to be a separating equilibrium, the best pooling policy, $(\bar{I}, 0) = (\bar{p}L, 0)$, where

$$\begin{aligned}\bar{p} &= \lambda p_L + (1 - \lambda)\, p^H \\ &= \frac{\lambda}{4} + \frac{1 - \lambda}{2} \\ &= \frac{2 - \lambda}{4},\end{aligned}$$

must not be preferred by a low risk individual.

The expected utility of the low risk type with the policy $(p^L(L-D^*), D^*)$ *is*

$$\begin{aligned} p^L u(W - p^L(L-D^*) - D^*) + \left(1-p^L\right) u(W - p^L(L-D^*)) &= \frac{1}{4}\ln\left(5 - \frac{1}{4}(4-\frac{10}{3}) - \frac{10}{3}\right) \\ &\quad +(1-\frac{1}{4})\ln\left(5 - \frac{1}{4}(4-\frac{10}{3})\right) \\ &= u_L^* \simeq 1.283. \end{aligned}$$

Hence we must have

$$u(W - \bar{p}L) \le u_L^*,$$

that is,

$$\ln\left(5 - \left(\frac{2-\lambda}{4}\right)4\right) = \ln(3+\lambda) \le u_L^*,$$

which may be written as

$$\lambda \le e^{u_L^*} - 3 \simeq 0.6.$$

(b) (10 points) If it is mandatory that policies offer full insurance, which policies will companies offer for each value of λ? Which individuals would subscribe them? (Again, show your solution graphically first, and then do the algebra.)

If full insurance is mandatory, then companies will offer the pooling policy $(\bar{I}, 0) = (\bar{p}L, 0)$ *if it is acceptable by the low risk individuals; that is, if*

$$u(W - \bar{p}L) \ge p^L u(W-L) + (1-p^L) u(W),$$

holds. Substituting, this inequality becomes

$$\ln(3+\lambda) \ge \frac{1}{4}\ln 1 + \frac{3}{4}\ln 5 = \frac{3}{4}\ln 5$$

or

$$\lambda \ge 5^{\frac{3}{4}} - 3 \simeq 0.34.$$

If $\lambda < 5^{\frac{3}{4}} - 3$*, then the insurance companies will offer the policy*

$$(I_H^*, D_H^*) = (p^H L, 0) = (2, 0)$$

and only high risk type individuals we get insurance.

Exercise 4.1. In a competitive labor there are workers of two types, H and L. The expected revenue of a firm that hires a worker of type $i \in \{H, L\}$ with a level of education $y \in \mathbb{R}_+$ is $\bar{x}(y,i) = a_i + \sqrt{y}$, where $a_H = 2$ and $a_L = 1$. Workers choose their levels of education before entering the labor market. Firms observe the workers' level of education, but not their types, and make a wage offer. The workers payoffs are $u_i(y,w) = w - c_i(y) - \underline{u}_i$, where c_i is the worker's cost of education, which is given by $c_H(y) = y/2$ and $c_L(y) = y$, and $\underline{u}_i$ is the worker's reservation utility, given by $\underline{u}_H = \underline{u}_L = 0$. Firms payoffs are their expected profits. (Of course, in a competitive equilibrium firms' profits are zero.)

(a) (10 points) Compute the most efficient separating PBNE (that is, the PBNE in which an worker's choice of y signals her type). (Show your solution graphically first, and then do the algebra.)

Solution.

In a competitive equilibrium $w_i = \bar{x}(y_i, i)$ *for* $i \in \{H, L\}$. *Note that* $y_L^* = 1/4$ *maximizes* $\bar{x}(y, L)$, *and* $y_H^* = 1$ *maximizes* $\bar{x}(y, H)$. *In the most efficient separating PBNE* $y_L = y_L^* = 1/4$ *and* $y_H := \bar{y}$ *is the smallest value in* $[0, \infty)$ *such that an* L *worker prefers to signal* y_L^* *than* y_H*; that is,*

$$w_L^* - c_L(y_L^*) \geq w_H^* - c_L(\bar{y}).$$

Substituting yields the inequality

$$(1 + \frac{1}{2}) - \frac{1}{4} \geq (2 + \sqrt{\bar{y}}) - \bar{y},$$

that is,

$$-\frac{3}{4} \geq \sqrt{\bar{y}} - \bar{y}.$$

Hence

$$\bar{y} = \frac{9}{4},$$

and

$$w_H = 2 + \sqrt{\frac{9}{4}} = \frac{7}{2}.$$

The equilibrium payoffs are

$$u_L^* = w_L^* - c_L(y_L^*) = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$$

and

$$u_H^* = w_H^* - c_H(\bar{y}) = \frac{7}{2} - \frac{9}{8} = \frac{19}{8}.$$

(b) (10 points) Assume that the fraction of workers of type L is $q \in (0,1)$. Compute the pooling PBNE in which the workers level of education $\hat{y}$ maximizes the surplus,

$$W(y) = q(\bar{x}(y,L) - c_L(y)) + (1-q)\,(\bar{x}(y,H) - c_H(y)),$$

and identify the values of q for which this equilibrium Pareto dominates the separating equilibrium identified in (a).

Solution. We have

$$\begin{aligned} W(y) &= q(1+\sqrt{y}-y) + (1-q)\,(2+\sqrt{y}-\frac{y}{2}) \\ &= 2-q+\sqrt{y}-\frac{1+q}{2}y, \end{aligned}$$

Hence

$$W'(y) = \frac{1}{2\sqrt{y}} - \frac{1+q}{2} = 0$$

yields

$$\hat{y}(q) = \frac{1}{(1+q)^2}.$$

The pooling wage is

$$\begin{aligned} \hat{w}(q) &= q\left(1+\frac{1}{1+q}\right) + (1-q)\left(2+\frac{1}{1+q}\right) \\ &= 2+\frac{1}{1+q}-q. \end{aligned}$$

Therefore

$$\hat{u}_L(q) = 2+\frac{1}{1+q}-q-\frac{1}{(1+q)^2}.$$

Obviously the low type is better of in this pooling equilibrium. (This is easy to show by noting that $d\hat{u}_L(q)/dq < 0$ on $(0,1)$ and $\hat{u}_L(1) = 5/4$.)

As for the high type, we have

$$\hat{u}_H(q) = 2+\frac{1}{1+q}-q-\frac{1}{2\,(1+q)^2} = \frac{19}{8}.$$

Therefore a high type worker is better off in the pooling equilibrium when

$$\hat{u}_H(q) > \frac{19}{8},$$

which requires

$$q \gtrapprox 0.12.$$

Exercise 4.2. (20 points) Calculate the seller's expected revenue and the (unique symmetric increasing differentiable) equilibrium bidding function $\beta(x)$ in a first-price sealed-bid action where there are 2 bidders whose values are independently and uniformly distributed on $[0,\omega]$. Assume that the seller is risk-neutral and his value for the object is zero. In this case the seller's expected utility is just his expected revenue. Calculate the optimal reserve price and the seller's expected revenue with this reserve price.

Solution. The cdf of a bidder's value X is $F(x) = x/\omega$. As seen in class (Proposition 2),

$$\beta(x) = \mathbb{E}(Y_1^{(1)} \big| Y_1^{(1)} < x).$$

The cdf of $Y_1^{(1)} \big| Y_1^{(1)} < x$ *is*

$$G(y) = \begin{cases} \dfrac{F(y)}{F(x)} = \dfrac{\frac{y}{\omega}}{\frac{x}{\omega}} = \dfrac{y}{x} & \text{if } y \leq x \\ 0 & \text{if } y > x. \end{cases} .$$

Hence

$$\beta(x) = \mathbb{E}(Y_1^{(1)} \big| Y_1^{(1)} < x) = \int_0^x y dG(y) = \frac{1}{x}\int_0^x y dy = \frac{x}{2}.$$

In order to calculate the seller's expected revenue we calculate a bidder's expected payment,

$$m(x) = G_1^{(1)}(x)\beta(x) = \left(\frac{x}{\omega}\right)\frac{x}{2} = \frac{x^2}{2\omega}.$$

Hence

$$\mathbb{E}(m(X)) = \int_0^\omega m(x)dF(x) = \int_0^\omega \frac{x^2}{2\omega}\left(\frac{1}{\omega}\right)dx = \frac{\omega}{6}.$$

The seller's expected revenue is

$$\bar{R} = n\mathbb{E}(m(X)) = \frac{\omega}{3}.$$

As seen in class the reserve price that maximizes the seller's expected payoff is the solution to the equation

$$r = \frac{1}{\lambda(r)},$$

where

$$\lambda(x) = \frac{f(x)}{1-F(x)} = \frac{\frac{1}{\omega}}{1-\frac{x}{\omega}} = \frac{1}{\omega - x}$$

is the hazard rate of F. Hence, in our setting this equation is $r = \omega - r$, i.e., $r^* = \omega/2$.

The seller's expected revenue is

$$\begin{aligned} \bar{R}(r^*) &= 2r^*\left(1-F(r^*)\right)F(r^*) + 2\int_{r^*}^\omega y\left(1-F(y)\right)dF(y) \\ &= 2\left(\frac{\omega}{2}\right)\left(1-\frac{\frac{\omega}{2}}{\omega}\right)\frac{\frac{\omega}{2}}{\omega} + 2\int_{\frac{\omega}{2}}^\omega y\left(1-\frac{y}{\omega}\right)\frac{dy}{\omega} \\ &= \frac{5\omega}{12}. \end{aligned}$$