Final Exam (June 2, 2017)

Exercise 1. Consider an economy that extends over two periods, today and tomorrow, in which there is a single perishable consumption good. The state of nature tomorrow can either be sunny (S) or cloudy (C). There are two consumers, A and B, whose preferences over consumption today (x), tomorrow when sunny (y), and tomorrow when cloudy (z) are represented by the utility functions $u^A(x,y,z) = x(y+2z)$ and $u^B(x,y,z) = x(2y+z)$, and whose endowments are $(\bar{x}_A, \bar{y}_A, \bar{z}_A) = (10, 20, 0)$, and $(\bar{x}_B, \bar{y}_B, \bar{z}_B) = (10, 0, 20)$, respectively.

(a) (25 points) Assume that there are contingent markets for all commodities. Denote by (p_x, p_y, p_z) the prices of x, y and z, respectively, and normalize $p_x = 1$. Calculate the competitive equilibrium prices and allocation. (Hint. Given x, the preferences of both agents are linear en y and z, that is, MRS^i_{yz} is a constant; specifically, $MRS^A_{yz} = -1/2$ and $MRS^B_{yz} = -2$. Therefore the optimal bundle involves $y = 0$ if $\left|MRS^i_{yz}\right| > p_y/p_z$, and $z = 0$ if $\left|MRS^i_{yz}\right| < p_y/p_z$. In the first case set $y = 0$ and solve for x and z, and in the second case set $z = 0$ and solve for x and y. Then use market clearing conditions to calculate the equilibrium prices (p^*_y, p^*_z).)

Solution: For $i \in \{A, B\}$, consumer i's problem is

$$\begin{array}{c} \max_{(x,y,z)\in\mathbb{R}^3_+} u^i(x,y,z) \\ \text{subject to: } x + p_y y + p_z z \leq \bar{x}^i + p_y \bar{y}^i + p_z \bar{z}^i. \end{array}$$

Consumer A: For prices (p_y, p_z), such that $p_y/p_z > 1/2$, $y_A = 0$, and (x_A, z_A) solves

$$\begin{array}{c} \max_{(x,y,z)\in\mathbb{R}^3_+} 2xz \\ \text{subject to: } x + p_y y + p_z z \leq 10 + 20p_y. \end{array}$$

Hence

$$\begin{aligned} x_A(p_y,p_z) &= \frac{10+20p_y}{2} = 5 + 10p_y \\ z_A(p_y,p_z) &= \frac{10+20p_y}{2p_z}. \end{aligned}$$

For prices (p_y, p_z), such that $p_y/p_z < 1/2$, $z_A = 0$, and (x_A, y_A) solves

$$\begin{array}{c} \max_{(x,y,z)\in\mathbb{R}^3_+} xy \\ \text{subject to: } x + p_y y + p_z z \leq 10 + 20p_y. \end{array}$$

Hence

$$\begin{aligned} x_A(p_y,p_z) &= \frac{10+20p_y}{2} = 5 + 10p_y \\ y_A(p_y,p_z) &= \frac{10+20p_y}{2p_y} = \frac{5}{p_y} + 10. \end{aligned}$$

Consumer B: Symmetrically, for $p_y/p_z < 2$, $z_B = 0$, *and*

$$\begin{aligned} x_B(p_y, p_z) &= \frac{10+20p_z}{2} = 5 + 10p_z \\ y_A(p_y, p_z) &= \frac{10+20p_z}{2p_y}, \end{aligned}$$

and for $p_y/p_z > 2$, $y_B = 0$, *and*

$$\begin{aligned} x_B(p_y, p_z) &= \frac{10+20p_z}{2} = 5 + 10p_z \\ y_A(p_y, p_z) &= \frac{10+20p_z}{2p_z} = \frac{5}{p_z} + 10. \end{aligned}$$

Market clearing conditions are

$$\begin{aligned} y_A(p_y, p_z) + y_B(p_y, p_z) &= \bar{y}_A + \bar{y}_B (= 20 + 0) \\ z_A(p_y, p_z) + z_B(p_y, p_z) &= \bar{z}_A + \bar{z}_B (= 0 + 20). \end{aligned}$$

Since $z_A = z_B = 0$ for $p_y/p_z < 1/2$, *and* $y_A = y_B = 0$ for $p_y/p_z > 2$, *then in a CE* $p_y/p_z \in [1/2, 2]$, *and therefore*

$$\begin{aligned} y_A(p_y, p_z) + y_B(p_y, p_z) &= 0 + \frac{10+20p_z}{2p_y} = 20 \\ z_A(p_y, p_z) + z_B(p_y, p_z) &= \frac{10+20p_y}{2p_z} + 0 = 20. \end{aligned}$$

Solving this system we get

$$(p_y^*, p_z^*) = (\frac{1}{2}, \frac{1}{2}).$$

The equilibrium allocation is

$$(x_A^*, y_A^*, z_A^*) = (10, 0, 20), \ (x_B^*, y_B^*, z_B^*) = (10, 20, 0).$$

Of course, this allocation is Pareto optimal by the First Welfare Theorem.

(b) (15 points) Suppose that there are no contingent markets, but there is a credit market and a market for a security that pays one unit of consumption tomorrow if sunny and 0 units of consumption if cloudy. Determine the competitive equilibrium interest rate r^* and security price q^*. (Hint. You need not repeat all calculations, but simple to explore the relation between (r^*, q^*) and the equilibrium prices found in part (a), (p_y^*, p_z^*).)

Solution: Let us normalize the spot prices to be $(\hat{p}_x, \hat{p}_y, \hat{p}_z) = (1, 1, 1)$*. For* (r, q)*, the problem of consumer is* $i \in \{A, B\}$

$$\begin{array}{l} \max_{[(x,y,x_C),b,y] \in \mathbb{R}^3_+ \times \mathbb{R} \times \mathbb{R}} u^i(x, y, z) \\ \text{subject to:} \\ \quad x + qs \leq \bar{x}^i + b \\ \quad y \leq y^i - (1+r)b + s \\ \quad z \leq \bar{z}^i - (1+r)b. \end{array}$$

Since consumers' utility functions are strictly increasing in all the arguments, the budget constraints are binding at the solution. Hence, solving for b *and* y *in the equation describing the constraints, we may write the problem as*

$$\begin{array}{c} \max_{(x,y,z) \in \mathbb{R}^3_+} u^i(x, y, z) \\ \text{subject to: } x + qy + (\frac{1}{1+r} - q)z \leq \bar{x}^i + q\bar{y}^i + (\frac{1}{1+r} - q)\bar{z}^i. \end{array}$$

This problem is identical to that of part (a). In equilibrium (r^*, q^*) *solves the system*

$$\begin{aligned} q &= p_y^* \\ \frac{1}{1+r} - q &= p_z^*. \end{aligned}$$

Solving the system we get

$$(q^*, r^*) = \left(\frac{1}{2}, 0\right).$$

And of course, the resulting allocation is that of part (a).

Exercise 2. The revenue of a risk-neutral principal is a random variable $X(e)$ taking values $x_1 = 2$ and $x_2 = 10$ with probabilities that depends on the level of effort of an agent, $e \in [0, 1]$, and are given by $p_1(e) = 1 - \sqrt{e}/2$ and $p_2(e) = \sqrt{e}/2$, respectively. There are two types of agents L and H with identical preferences represented by the von Neumann-Morgenstern utility function $u(w) = \sqrt{w}$, and identical reservation utility $\underline{u} = 0$, but different costs of effort given by $v_L(e) = e$ and $v_H(e) = 2e$.

(a) (10 points) Assume that *effort is verifiable* and the Principal observes the agent's type. Determine the contracts the principal will offer to each type of agent. Illustrate your results providing a graph the effort supply and effort demand functions for each type of agent.

Solution. For $e \in [0, 1]$,

$$E[X(e)] = 2\left(1 - \frac{\sqrt{e}}{2}\right) + 10\left(\frac{\sqrt{e}}{2}\right) = 2 + 4\sqrt{e}.$$

For $\tau \in \{H, L\}$, *the principal's problem is*

$$\begin{aligned} \max_{(e,w)\in[0,1]\times\mathbb{R}_+} \quad & 2 + 4\sqrt{e} - w \\ s.t. \quad & \sqrt{w} \geq K^\tau e, \end{aligned}$$

where $K^H = 2$, *and* $K^L = 1$. *The first order conditions for an interior solution are described by the system of equations*

$$\begin{aligned} \frac{4}{2\sqrt{e}} &= 2K^\tau \sqrt{w} \\ \sqrt{w} &= K^\tau e. \end{aligned}$$

The first equation defines the Principal's demand of effort, and the second equation defines the Agent's supply of effort.

For $\tau = L$ *these functions are*

$$\begin{aligned} w &= \frac{1}{e} \\ w &= e^2. \end{aligned}$$

Hence the optimal contract is

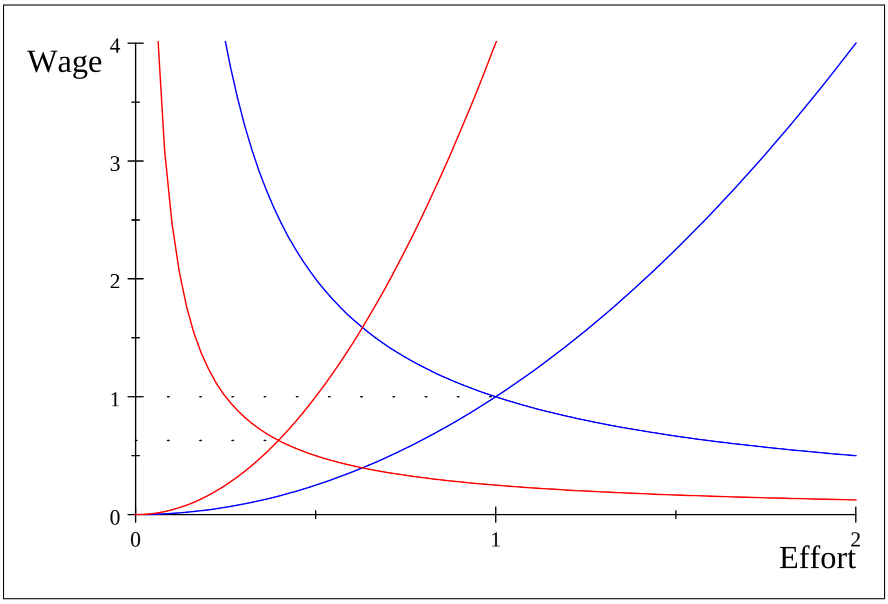
$$(e^L, w^L) = (1, 1).$$

For $\tau = H$ these functions are

$$\begin{aligned} w &= \frac{1}{4e} \\ w &= 4e^2. \end{aligned}$$

Hence the optimal contract is

$$(e^H, w^H) = (\frac{1}{16^{\frac{1}{3}}}, \frac{4}{16^{\frac{2}{3}}}).$$



(b) (15 points) Now assume that the Agent's type is observable, but *effort is not verifiable.* Also assume that only two efforts levels are feasible, $e = 1/4$ and $e = 1$. Determine the contracts the Principal will offer to each type of agent.

Solution. Since upon accepting a contract the lowest effort an agent can exert is $e = 1/4$, the optimal contracts for inducing agents to exert low effort, $e = 1/4$, involve fixed wages satisfying the participations constrains

$$\sqrt{w^{\tau}} = K^{\tau}\left(\frac{1}{4}\right);$$

that is, $\bar{w}^{L} = 1/16$, and $\bar{w}^{H} = 1/4$. The resulting profit for $\tau \in \{H, L\}$ are

$$E\left[X(\frac{1}{4})\right] - w^{\tau} = 2 + 4\sqrt{\frac{1}{4}} - \bar{w}^{\tau} = 4 - \bar{w}^{\tau} > 0.$$

The incentive compatible contract for the agents of type L to exert high effort ($e = 1$) solves the system $\left(\frac{1}{2}\right)x - 1 = \frac{1}{4}x - \frac{1}{4}$, Solution is: 3

$$\begin{aligned}\left(1-\frac{1}{2}\right)\sqrt{w_1} + \left(\frac{1}{2}\right)\sqrt{w_2} &= 1\\ \left(1-\frac{1}{2}\right)\sqrt{w_1} + \left(\frac{1}{2}\right)\sqrt{w_2} - 1 &= \frac{3}{4}\sqrt{w_1} + \frac{1}{4}\sqrt{w_2} - \frac{1}{4}.\end{aligned}$$

The solution to this system involves a negative wage w_1. Assuming that negative wages cannot be paid due to limited liability, forces the Principal to set up $w_1 = 0$, and hence the incentive compatibility constraint implies $w_2 = 3$. For this wage contract, $(w_1^L, w_2^L) = (0, 3)$, the Principal's profit is

$$E\left[X(1)\right] - \frac{1}{2}(3) = 6 - \frac{3}{2} = 4.5 > 4 - w^{L} = \frac{63}{16}.$$

Hence the optimal to offer the agents of type L is $(\tilde{e}^L, \tilde{w}_1^L, \tilde{w}_2^L) = (1, 0, 3)$.

The incentive compatible contract for the agents of type H to exert high effort solves the system

$$\begin{aligned}\left(1-\frac{1}{2}\right)\sqrt{w_1} + \left(\frac{1}{2}\right)\sqrt{w_2} &= 2\\ \left(1-\frac{1}{2}\right)\sqrt{w_1} + \left(\frac{1}{2}\right)\sqrt{w_2} - 2 &= \frac{3}{4}\sqrt{w_1} + \frac{1}{4}\sqrt{w_2} - \frac{2}{4},\end{aligned}$$

Again the solution to this system involves a negative wage w_1. Setting $w_1 = 0$, requires $w_2 = 6$ in order to satisfy the incentive compatibility constraint. The Principal's profit with this wage contract $(w_1^H, w_2^H) = (0, 6)$ is

$$E\left[X(1)\right] - \frac{1}{2}(6) = 6 - 3 < 4 - \bar{w}^{H} = \frac{15}{4}.$$

Hence the optimal contract to offer the agents of type H is $(\tilde{e}^H, \tilde{w}_1^H, \tilde{w}_2^H) = (1/4, 1/4, 1/4)$.

(c) (15 points) Now assume that effort is verifiable, and that only two efforts levels, $e = 1/4$ and $e = 1$, are feasible. However, the Principal *does not observe the agent's type.* Agents of type H and L are present in the population of agents in fractions $q \in (0, 1)$ and $1-q$, respectively. Identify the Principal's optimal menu of contracts for each value of q. (Keep on mind that the Principal may choose to offer a single contract, which may be acceptable either both types or only by the low cost type, if either of these contracts generates more profit than the optimal menu of contracts satisfying participation and incentive constraints.)

Solution. The Principal may offer a single "pooling" contract, which can be either the contract $(e, w) = (1/4, 1/4)$*, which both agents accept, leading to an expected profit of*

$$\Pi_H = E\left[X(\frac{1}{4})\right] - \frac{1}{4} = 4 - \frac{1}{4} = \frac{64}{16},$$

or the contract $(1, 1)$*, which only the agents of type L accept, leading to the an expected profit of*

$$\Pi_L = (1-q)\left(E[X(1)] - 1\right) = 5\,(1-q)\,.$$

The Principal may also design an incentive compatible menu of contracts involving low effort for the high type, $e^H = 1/4$*, and high effort for the low type,* $e^L = 1$*. As shown in class, such menu involve wages* w^L *and* w^H *that are identified by participation constraint of the type* H *and the incentive of the type* L*,*

$$\begin{aligned} \sqrt{w^H} &\geq 2e^H \qquad (PC_H) \\ \sqrt{w^L} - e^L &\geq \sqrt{w^H} - e^H \qquad (IC_L). \end{aligned}$$

That is,

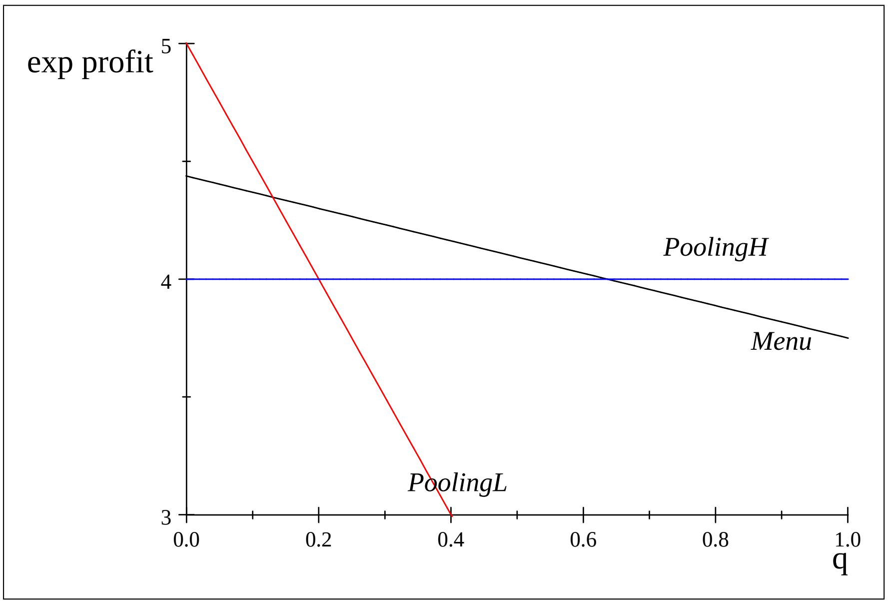
$$\begin{aligned} \sqrt{w^H} &= 2\left(\frac{1}{4}\right) \\ \sqrt{w^L} - 1 &= \sqrt{\tilde{w}^H} - \frac{1}{4}. \end{aligned}$$

whose solution is $w^H = 1/4$ *and*

$$w^L = \left(1 + \left(\sqrt{\frac{1}{4}} - \frac{1}{4}\right)\right)^2 = \frac{25}{16}.$$

For this menu of contracts, $\{(1/4, 1/4), (1, 25/16)\}$*, the expected profit is*

$$\begin{aligned} \Pi_S &= q\left(E[X(\frac{1}{4})] - \frac{1}{4}\right) + (1-q)\left(E[X(1)] - \frac{25}{16}\right) \\ &= q\left(4 - \frac{1}{4}\right) + (1-q)\left(6 - \frac{25}{16}\right) \\ &= \frac{71}{16} - \frac{11}{16}q. \end{aligned}$$



Thus, for low values of q it is optimal to offer the contract $(1,1)$. Specifically, for q such that

$$5(1-q) > \frac{71}{16} - \frac{11}{16}q \Leftrightarrow q < \frac{3}{23}.$$

For high values of q it is optimal to offer the contract $(1/4, 1/4)$. Specifically, for q such that

$$\frac{71}{16} - \frac{11}{16}q < \frac{64}{16} \Leftrightarrow q > \frac{7}{11}.$$

For intermediate values of q, that is, for $q \in (\frac{3}{23}.\frac{7}{11})$, offering the menu $\{(1/4, 1/4), (1, 25/16)\}$ is optimal.

Exercise 3A. A small town must decide the number of hours of street cleaning service x it will have. The cost of street cleaning services is 3 euros/hour. Each citizen $i \in \{1,2,3\}$ is endowed with $\bar{y} = 10$ euros, and her preferences are described by a utility function of the form $u^i(x,y) = y + a_i\sqrt{x}$, where y denotes income (in euros) available to spend on other goods, and $a_i > 0$ measures citizen i's *intensity* of preference for street cleaning service. Assume that $a_1 = 2$, $a_2 = 4$, and $a_3 = 6$

(a) (10 points) Identify the conditions that characterize interior Pareto optimal, and Lindahl equilibrium allocations for this economy.

Solution. A Pareto optimal allocation (x, y_1, y_2, y_3) is a solution to the system:

$$\begin{aligned} |MRS_1(x,y)| + |MRS_2(x,y)| + |MRS_3(x,y)| &= 3 \\ y_1 + y_2 + y_3 + 3x &= 3\bar{y} \end{aligned}$$

Since for $i \in \{1,2,3\}$,

$$MRS_i(x,y) = -\frac{\partial u^A/\partial x}{\partial u^A/\partial y} = -\frac{a_i}{2\sqrt{x}},$$

this system becomes

$$\begin{aligned} \frac{2}{2\sqrt{x}} + \frac{4}{2\sqrt{x}} + \frac{6}{2\sqrt{x}} &= 3 \\ y_1 + y_2 + y_3 + 3x &= 30. \end{aligned}$$

The first equation determines the optimal level of public good,

$$\frac{6}{\sqrt{x}} = 3 \Rightarrow x = 4.$$

Thus, any allocation (x, y_1, y_2, y_3) such that $x = 4$ and $y_1 + y_2 + y_3 = 18$ is Pareto optimal.

In a Lindahl equilibrium the system of personalized prices must be such that for $i \in \{1,2,3\}$

$$|MRS_i(x,y)| = \frac{a_i}{2\sqrt{x}} = p_i$$

must hold for $x = 4$. Hence $p_1 = \frac{1}{2}$, $p_1 = 1$, and $p_3 = \frac{3}{2}$. Thus, the Lindahl allocation is $(x^L, y_1^L, y_2^L, y_3^L) = (4, 8, 6, 4)$, and individual utilities are $(u_1^L, u_2^L, u_3^3) = (12, 14, 18)$.

(b) (10 points) Calculate the number of hours of street cleaning service under voluntary contribution, and determine whether the resulting allocation is Pareto optimal. Is the Lindahl equilibrium Pareto superior to this allocation?

Solution. Under voluntary contribution, the contribution of individual $i \in \{1, 2, 3\}$, $z_i \in R_+$, *solves the problem*

$$\max_{z_i \in \mathbb{R}_+} y + a_i \sqrt{\frac{z_{-i} + z_i}{3}}$$

$$\text{subject to:}$$

$$y + z_i = 10,$$

where z_{-i} *is the sum of the contributions of individuals other than* i*. This problem is equivalent to*

$$\max_{z_i \in \mathbb{R}_+} (10 - z_i) + a_i \sqrt{\frac{z_{-i} + z_i}{3}}.$$

The first order condition for a solution to this problem is

$$\frac{a_i}{2\sqrt{z_{-i} + z_i}} = \sqrt{3}$$

That is

$$z_i = \max\{\frac{a_i^2}{12} - z_{-i}, 0\}.$$

Since $a_3 = 6 > a_2 = 4 > a_1 = 2$*, in equilibrium*

$$z_3^* + z_{-3}^* \geq \frac{a_3^2}{12} = 3.$$

Therefore $z_1^* = z_2^* = 0$*, and* $z_3^* = 3$*. Hence under voluntary contribution the level of street cleaning is*

$$x^* = \frac{z_1^* + z_2^* + z_3^*}{3} = 1,$$

which is suboptimal, since as shown in part (a) a Pareto optimal allocation involves $x = 4$*.*

Computing agents' utilities in this allocation, given by $(x^V, y_1^V, y_2^V, y_3^V) = (1, 10, 10, 7)$*, we get* $(u_1^V, u_2^V, u_3^V) = (12, 14, 13)$*. Thus, individuals 1 and 2 are equally well off, whereas individual 3 is worse off, than in the Lindahl equilibrium. Hence the Lindahl equilibrium is, in this example, Pareto superior to the allocation generated by voluntary contribution. (This result does not hold generally.)*

Exercise 3B. In any given day, tourists traveling to certain city known to be a pickpocket's playground face the risk of loosing the 32 euros they typically carry in their wallet. For the more alert tourists, this happens with probability $p_L = 1/4$, while for the inattentive ones this probability is $p_H = 1/2$. Each tourist has an allowance of $W = 100$ euros for the day, and his preferences are described by the von Neumann-Morgenstern utility function $u(x) = \ln x$. It is known that half of the tourist are alert, and the other half are inattentive. Insurance companies cannot distinguish amongst tourists of either type.

(a) (10 points) If there is a competitive insurance market, which insurance policies will be offered?

Solution. As established in class, a competitive equilibrium, when it exists, offers separating fair policies $(I_H, 0) = (16, 0)$ *and* $(\hat{I}_L, \hat{D}_L)$ *such that*

$$\hat{I}_L = (32 - \hat{D}_L)p_L$$

and the inattentive tourist will be indifferent between the two policies, i.e.,

$$\frac{1}{2}\ln\left(100 - (32 - \hat{D}_L)p_L - \hat{D}_L\right) + \frac{1}{2}\ln\left(100 - (32 - \hat{D}_L)p_L\right) = \ln(100 - 16).$$

This equation may be write for $x = D_L$ *as*

$$(100 - (32 - x)/4 - x)\,(100 - (32 - x)/4) = (100 - 16)^2,$$

that is

$$-\frac{3}{16}x^2 - 46x + 1408 = 0.$$

Solving this equation we get $\hat{D}_L = \frac{16}{3}\sqrt{793} - \frac{368}{3} \simeq 27.521$.

For these policies to form a competitive equilibrium the alert tourist must prefer the policy $(\hat{I}_L, \hat{D}_L)$ *to the pooling policy* $(\bar{I}, 0) = (32\bar{p}, 0)$, *where*

$$\bar{p} = \frac{1}{2}p_H + \frac{1}{2}p_L = \frac{3}{8}.$$

That is $(\bar{I}, 0) = (12, 0)$.

Since the expected utility of an alert tourist with the separating policy is

$$\begin{aligned}
&\frac{1}{4}\ln\left(100 - \left(32 - \left(\frac{16}{3}\sqrt{793} - \frac{368}{3}\right)\right)/4 - \left(\frac{16}{3}\sqrt{793} - \frac{368}{3}\right)\right) \\
&+\frac{3}{4}\ln\left(100 - \left(\frac{16}{3}\sqrt{793} - \frac{368}{3}\right)/4\right) \\
\simeq\ & 4.467,
\end{aligned}$$

and his expected utility with the pooling policy is

$$\ln(100 - 12) \simeq 4.477,$$

there is no competitive equilibrium in this market.

(b) (10 points) If the market is monopolized by a single company which cannot discriminate amongst tourists by law (i.e., must offer a single policy), which policy will be offered? (Hint. Should the monopoly offer full insurance? Should the monopoly offer a policy intended for both types of tourists, or one that only inattentive tourist would subscribe?)

Solution. The company must decide with to offer a policy that only inattentive tourist subscribe or one which both types of tourists subscribe Obviously, in either case the company will offer full insurance since it can extract more surplus from the risk averse tourists. The largest premium the inattentive tourists are will to pay solves the equation

$$\ln(100 - x) = \frac{1}{2}\ln(100 - 32) + \frac{1}{2}\ln(100),$$

that is,

$$(100 - x)^2 - (100 - 32)(100) = x^2 - 200x + 3200 = 0.$$

Solving this equation we get $I_L^* = 100 - 20\sqrt{17} \simeq 17.538$, *and expected profits are*

$$\frac{1}{2}\left(100 - 20\sqrt{17} - 32p_H\right) = \frac{1}{2}\left(100 - 20\sqrt{17} - 16\right) \simeq 0.76894.$$

If the firm offers a policy that both types subscribe, it has to offer it at the maximum premium the alert tourist are willing to pay, that is,

$$\ln(100 - x) = \frac{1}{4}\ln(100 - 32) + \frac{3}{4}\ln(100),$$

or

$$(100 - x)^4 - (100 - 32)(100)^3 = (100 - x)^4 - 68000000 = 0,$$

that is

$$\bar{I}^* = 100 - \sqrt[4]{68000000} \simeq 9.1913$$

The expected profits offering this policy are

$$\bar{I}^* - 32\bar{p} = \bar{I}^* - 12 < 0.$$

Hence the company will offer the policy $(I_L^*, 0)$, *which will be subscribed only by inattentive tourists.*