CHAPTER 4: Higher order derivatives

4-1. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by $u(x,y) = e^x \sin y$. Find all the second partial derivatives D^2u, and verify Schwarz's Theorem.

4-2. Consider the quadratic function $Q : \mathbb{R}^3 \to \mathbb{R}$ defined by $Q(x,y,z) = x^2 + 5y^2 + 4xy - 2yz$. Compute the Hessian matrix D^2Q.

4-3. Let $f(x,y,z) = e^z + \frac{1}{x} + xe^{-y}$, for $x \neq 0$. Compute

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}$$

4-4. Let $z = f(x,y)$, $x = at$, $y = bt$ where a and b are constant. Consider z as a function of t. Compute $\frac{d^2 z}{dt^2}$ in terms of a, b and the second partial derivatives of f: f_{xx}, f_{yy} and f_{xy}.

4-5. Let $f(x,y) = 3x^2y + 4x^3y^4 - 7x^9y^4$. Compute the Hessian matrix D^2Q..

4-6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two functions whose partial derivatives are continuous on all of $\mathbb{R}^2$ and such that there is a function n $h : \mathbb{R}^2 \to \mathbb{R}$ such that $(f,g) = \nabla h$, that is,

$$f(x,y) = \frac{\partial h}{\partial x}(x,y) \quad g(x,y) = \frac{\partial h}{\partial y}(x,y)$$

at every point $(x,y) \in \mathbb{R}^2$. What equation do

$$\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial g}{\partial x}$$

satisfy?

4-7. The demand function of a consumer by a system of equations of the form

$$\begin{aligned} \frac{\partial u}{\partial x} &= \lambda p_1 \\ \frac{\partial u}{\partial y} &= \lambda p_2 \\ p_1 x + p_2 y &= m \end{aligned}$$

where $u(x,y)$ is the utility function of the agent, p_1 and p_2 are th prices of the consumption bundles, m is income and $\lambda \in \mathbb{R}$. Assuming that this system determines x, y and λ as functions of the other parameters, determine

$$\frac{\partial x}{\partial p_1}$$

4-8. Consider the system of equations

$$\begin{aligned} z^2 + t - xy &= 0 \\ zt + x^2 &= y^2 \end{aligned}$$

(a) Prove that it determines z and t as functions of x, y near the point $(1,0,1,-1)$.
(b) Compute the partial derivatives of z and t with respect to x, y at $(1,0)$.
(c) Without solving the system, ¿what is approximate value of $z(1'001, 0'002)$
(d) Compute

$$\frac{\partial^2 z}{\partial x \partial y}(1,0)$$

4-9. Consider the system of equations

$$\begin{aligned} xt^3 + z - y^2 &= 0 \\ 4zt &= x - 4 \end{aligned}$$

(a) Prove that it determines z and t as functions of x, y near the point $(0, 1, 1, -1)$.
(b) Compute the partial derivatives of z and t with respect to x, y at $(0, 1)$.
(c) Without solving the system, ¿what is approximate value of $z(0'001, 1'002)$
(d) Compute

$$\frac{\partial^2 z}{\partial x \partial y}(0, 1)$$

4-10. Find the second order Taylor polinomial for the following functions about the given point.
(a) $f(x, y) = \ln(1 + x + 2y)$ about the point $(2, 1)$.
(b) $f(x, y) = x^3 + 3x^2y + 6xy^2 - 5x^2 + 3xy^2$ about the point $(1, 2)$.
(c) $f(x, y) = e^{x+y}$ about the point $(0, 0)$.
(d) $f(x, y) = \sin(xy) + \cos(xy)$ about the point $(0, 0)$.
(e) $f(x, y, z) = x - y^2 + xz$ about the point $(1, 0, 3)$.

4-11. For what values of the parameter a is the quadratic form $Q(x, y, z) = x^2 - 2axy - 2xz + y^2 + 4yz + 5z^2$ positive definite?

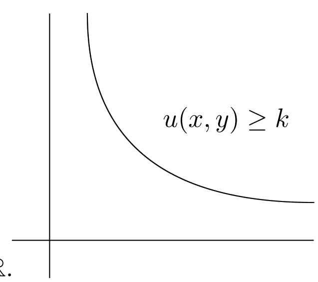
4-12. Study the signature of the following quadratic forms.
(a) $Q_1(x, y, z) = x^2 + 7y^2 + 8z^2 - 6xy + 4xz - 10yz$.
(b) $Q_2(x, y, z) = -2y^2 - z^2 + 2xy + 2xz + 4yz$.

4-13. Study for what values of a the quadratic form $Q(x, y, z) = ax^2 + 4ay^2 + 4az^2 + 4xy + 2axz + 4yz$ is
(a) positive definite.
(b) negative definite.

4-14. Classify the following quadratic forms, depending on the parameters.

$$\text{a) } Q(x, y, z) = 9x^2 + 3y^2 + z^2 + 2axz$$
$$\text{b) } Q(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + bx_3^2 + 2ax_1x_2 + 2x_2x_3$$

4-15. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a concave function so that for every $v_1, v_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have that $u(\lambda v_1 + (1 - \lambda)v_2) \geq \lambda u(v_1) + (1 - \lambda)u(v_2)$. Show that $S = \{v \in \mathbb{R}^n : u(v) \geq k\}$ is a convex set. For a concave $u : \mathbb{R}^2 \to \mathbb{R}$, the figure represents its graph $S = \{(x, y) \in \mathbb{R}^2 : u(x, y) \geq k\}$



4-16. State the previous problem for a convex function $u : \mathbb{R}^n \to \mathbb{R}$.

4-17. Determine the domains of the plane where the following functions are convex or concave.
(a) $f(x, y) = (x - 1)^2 + xy^2$.
(b) $g(x, y) = \frac{x^3}{3} - 4xy + 12x + y^2$.
(c) $h(x, y) = e^{-x} + e^{-y}$.
(d) $k(x, y) = e^{xy}$.
(e) $l(x, y) = \ln \sqrt{xy}$.

4-18. Determine the values of the parameters a and b so that the following functions are convex in their domains.
(a) $f(x, y, z) = ax^2 + y^2 + 2z^2 - 4axy + 2yz$
(b) $g(x, y) = 4ax^2 + 8xy + by^2$

4-19. Discuss the concavity and convexity of the function $f(x, y) = -6x^2 + (2a + 4)xy - y^2 + 4ay$ according to the values of a.

4-20. Find the largest convex set of the plane where the function $f(x, y) = x^2 - y^2 - xy - x^3$ is concave.