CHAPTER 3: Partial derivatives and differentiation

3-1. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ for the following functions:
(a) $f(x, y) = x \cos x \sin y$.
(b) $f(x, y) = e^{xy^2}$.
(c) $f(x, y) = (x^2 + y^2) \ln(x^2 + y^2)$.

3-2. Determine the marginal-products for the following production function.

$$F(x, y, z) = 12x^{1/2}y^{1/3}z^{1/4}$$

3-3. Find the gradient of the following functions at the given point p
(a) $f(x, y) = \left(a^2 - x^2 - y^2\right)^{1/2}$ at $p = (a/2, a/2)$.
(b) $g(x, y) = \ln(1 + xy)^{1/2}$ at $p = (1, 1)$.
(c) $h(x, y) = e^y \cos(3x + y)$ at $p = (2\pi/3, 0)$.

3-4. Consider the function

$$f(x, y) = \begin{cases} 2\frac{x^2+y^2}{|x|+|y|} \sin(xy) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Find the partial derivatives of f at the point $(0, 0)$.
(b) Prove that f is continuous on all of $\mathbb{R}^2$. *Hint:* Use (proving it) that for $(x, y) \neq (0, 0)$ we have that

$$0 \leq \frac{\sqrt{x^2 + y^2}}{|x| + |y|} \leq 1$$

(c) Is f differentiable at $(0, 0)$?

3-5. Consider the function

$$f(x, y) = \begin{cases} \frac{x \sin y}{x^2+y^2} & if (x, y) \neq (0, 0) \\ 0 & if (x, y) = (0, 0) \end{cases}$$

(a) Study the continuity of f in $\mathbb{R}^2$.
(b) Compute the partial derivatives of f at the point $(0, 0)$.
(c) At which points is f differentiable?

3-6. Consider the function

$$f(x, y) = \begin{cases} 2\frac{x^3 y}{x^2+2y^2} \cos(xy) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Find the partial derivatives of f at the point $(0, 0)$.
(b) Prove that f is continuous on all of $\mathbb{R}^2$. *Hint:* Note that for $(x, y) \neq (0, 0)$ we have that

$$\frac{1}{x^2 + 2y^2} \leq \frac{1}{x^2 + y^2}$$

(c) Is f differentiable at $(0, 0)$?

3-7. Compute the derivatives of the following functions at the given point p along the vector v
(a) $f(x, y) = x + 2xy - 3y^2$, $p = (1, 2)$, $v = (3, 4)$.
(b) $g(x, y) = e^{xy} + y \tan^{-1} x$, $p = (1, 1)$, $v = (1, -1)$.
(c) $h(x, y) = \left(x^2 + y^2\right)^{1/2}$, $p = (0, 5)$, $v = (1, -1)$.

3-8. Let $B(x, y) = 10x - x^2 - \frac{1}{2}xy + 5y$ be the profits of a firm. Last year the company sold $x = 4$ units of good 1 and $y = 2$ units of good 2. This year, the company can change slightly the amounts of the goods x and y it sells. If it wishes to increase its profit as much as possible, what should $\frac{\Delta x}{\Delta y}$ be?

3-9. Knowing that $\frac{\partial f}{\partial x}(2,3) = 7$ and $D_{(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})}f_{(2,3)} = 3\sqrt{5}$, find $\frac{\partial f}{\partial y}(2,3)$ and $D_v f_{(2,3)}$ with $v = (\frac{3}{5}, \frac{4}{5})$.

3-10. Find the derivative of $f(x,y,z) = xy^2 + z^2 y$, along the vector $v = (1,-1,2)$ at the point $(1,1,0)$. Determine the direction which maximizes (resp. minimizes) the directional derivative at the point $(1,1,0)$. What are the largest and smallest values of the directional derivative at that point?

3-11. Consider the function $f(x,y) = x^2 + y^2 + 1$ y $g(x,y) = (x+y, ay)$. Determine:
(a) The value of a for which the function $f \circ g$ grows fastest in the direction of the vector $v = (5,7)$ at the point $p = (1,1)$.
(b) The equations of the tangent and normal lines to the curve $xy^2 - 2x^2 + y + 5x = 6$ at the point $(4,2)$.

3-12. Find the Jacobian matrix of F in the following cases.
(a) $F(x,y,z) = (xyz, x^2 z)$
(b) $F(x,y) = (e^{xy}, \ln x)$
(c) $F(x,y,z) = (\sin xyz, xz)$

3-13. Using the chain rule compute the derivatives

$$\frac{\partial z}{\partial r} \quad \frac{\partial z}{\partial \theta}$$

in the following cases.
(a) $z = x^2 - 2xy + y^2$, $x = r + \theta$, $y = r - \theta$
(b) $z = \sqrt{25 - 5x^2 - 5y^2}$, $x = r\cos\theta$, $y = r\sin\theta$

3-14. Using the capital K at time t generates an instant profit of

$$B(t) = 5(1+t)^{1/2} K$$

Suppose that capital evolves in time according to the equation $K(t) = 120e^{t/4}$. Determine the rate of change of B.

3-15. Verify the chain rule for the function $h = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ with $x = e^t$, $y = e^{t^2}$ and $z = e^{t^3}$.

3-16. Verify the chain rule for the composition $f \circ c$ in the following cases.
(a) $f(x,y) = xy$, $c(t) = (e^t, \cos t)$.
(b) $f(x,y) = e^{xy}$, $c(t) = (3t^2, t^3)$.

3-17. Write the chain rule $h'(x)$ in the following cases.
(a) $h(x) = f(x, u(x,a))$, where $a \in \mathbb{R}$ is a parameter.
(b) $h(x) = f(x, u(x), v(x))$.

3-18. Determine the points at which the tangent plane to the surface $z = e^{(x-1)^2 + y^2}$ is horizontal. Determine the equation of the tangent plane at those points.

3-19. Consider the function $f(x,y) = (xe^y)^3$.
(a) Compute the equation of the tangent plane to the graph of $f(x,y)$ at the point $(2,0)$.
(b) Using the equation of the tangent plane, find an approximation to $(1{,}999e^{0{,}002})^3$.

3-20. Compute the tangent plane and normal line to the following level surfaces.
(a) $x^2 + 2xy + 2y^2 - z = 0$ at the point $(1,1,5)$.
(b) $x^2 + y^2 - z = 0$ at the point $(1,2,5)$.
(c) $(y - x^2)(y - 2x^2) - z = 0$ at the point $(1,3,2)$.

3-21. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two functions with continuous partial derivatives on $\mathbb{R}^2$.
(a) Show that if

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial g}{\partial x}(x,y)$$

at every point $(x,y) \in \mathbb{R}^2$, then $f - g$ depends only on y.
(b) Show that if

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial g}{\partial y}(x,y)$$

at every point $(x, y) \in \mathbb{R}^2$, then $f - g$ depends only on x.

(c) Show that if $\nabla(f - g)(x, y) = (0, 0)$ at every point $(x, y) \in \mathbb{R}^2$, then $f - g$ is constant on $\mathbb{R}^2$.

(d) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial y}(x, y) = yx^2 + x + 2y, \quad \frac{\partial f}{\partial x}(x, y) = y^2 x + y, \quad f(0, 0) = 1$$

Are there any other functions satisfying those equations?