UNIVERSITY CARLOS III OF MADRID

MATHEMATICS FOR ECONOMICS I FALL 2021

EXERCISES (SOLUTIONS )

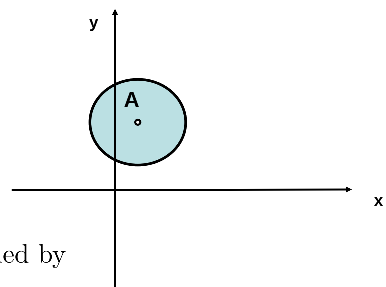
## CHAPTER 1: Introduction to the Topology of Euclidean Space $\mathbb{R}^n$.

2-1. *Sketch the following subsets of $\mathbb{R}^2$. Sketch their boundary and the interior. Study whether the following are closed, open, bounded and/or convex.*

(a) $A = \{(x, y) \in \mathbb{R}^2 : 0 < \|(x, y) - (1, 3)\| < 2\}$.
(b) $B = \{(x, y) \in \mathbb{R}^2 : y \leq x^3\}$.
(c) $C = \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| \leq 2\}$.
(d) $D = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$.
(e) $E = \{(x, y) \in \mathbb{R}^2 : y < x^2, y < 1/x, x > 0\}$.
(f) $F = \{(x, y) \in \mathbb{R}^2 : xy \leq y + 1\}$.
(g) $G = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 \leq 1, x \leq 1\}$.

**Solution:**

(a) The set represents the disk of center $C = (1, 3)$ and radius 2 with the center removed.




The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \|(x, y) - (1, 3)\| = \sqrt{(x - 1)^2 + (y - 3)^2}$$

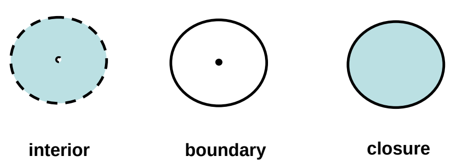
is continuous and the set $A$ may be written as

$$A = \{(x, y) \in \mathbb{R}^2 : 0 < f(x, y) < 2\} = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in (0, 2)\}$$

Since, the interval $(0, 2) \subset \mathbb{R}$ is open, the set $A$ is **open**. It is also **bounded**, since it is contained in the disc $\{(x, y) \in \mathbb{R}^2 : \|(x, y) - (1, 3)\| < 2\}$.
In addition, **it is not convex** since the points $P = (1, 4)$ and $Q = (1, 2)$ belong to $A$ but the convex combination

$$\frac{1}{2}(1, 4) + \frac{1}{2}(1, 2) = (1, 3)$$

does not belong to the set $A$.
The interior, boundary and closure of $A$ are represented in the following figure




Note that $\partial A \cap A = \emptyset$. This gives another way to prove that the set $A$ is open.

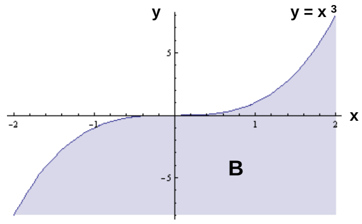
(b) The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^3 - y$$

is continuous and the set $B$ may be written as

$$B = \{(x, y) \in \mathbb{R}^2 : f(x, y) \geq 0\} = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in [0, \infty)\}$$

Since, the interval $[0, \infty) \subset \mathbb{R}$ is closed, the set $B$ is **closed**.




The set $B$ **is not bounded** since, for example, the points

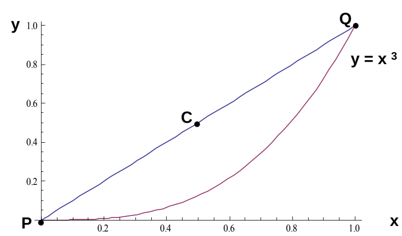
$$(1,0),(2,0),\ldots,(n,0),\ldots$$

belong to $B$ and

$$\lim_{n\to\infty} \|(n,0)\| = +\infty$$

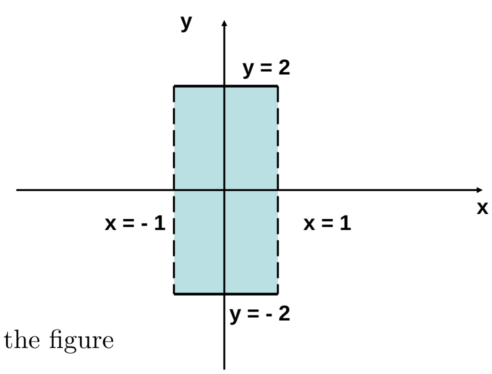
Furthermore, **it is not convex** since the points $P=(0,0)$ and $Q=(1,1)$ belong to $B$ but the convex combination

$$C = \frac{1}{2}P + \frac{1}{2}Q = \left(\frac{1}{2},\frac{1}{2}\right)$$

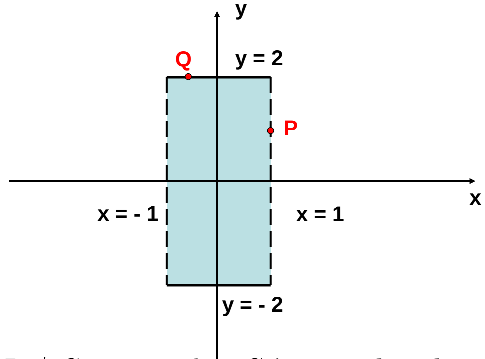



does not belong to $B$, because it does not satisfy the equation $y \le x^3$.
The interior of $B$ is the set $\{(x,y)\in\mathbb{R}^2 : y < x^3\}$. The boundary of $B$ is the set $\partial(B) = \{(x,y)\in\mathbb{R}^2 : y = x^3\}$. And the closure of $B$ is the set $\bar{B} = B\cup\partial(B) = \{(x,y)\in\mathbb{R}^2 : y \le x^3\}$. Since, $\bar{B} = B$, the set **is closed**.

(c) Graphically, the set $C$ is



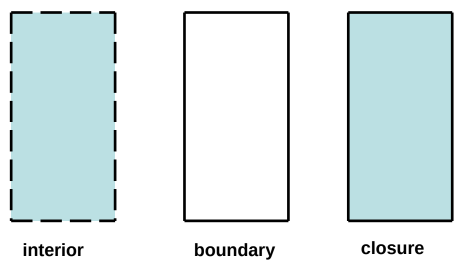

The points $P$ and $Q$ in the figure




belong to $\partial(C)$. Since, $P \notin C$, we see that $C$ **is not closed** and since $Q \in C$, we see that $C$ **is not open**.
Graphically, we see that the set $C$ **is convex**. An alternative way to prove this is by noting that the set $C$ is determined by the following **linear inequalities**

$$x > -1, \qquad x < 1, \qquad y \geq -2, \qquad y \leq 2$$

The interior, boundary and closure of $A$ are represented in the following figure

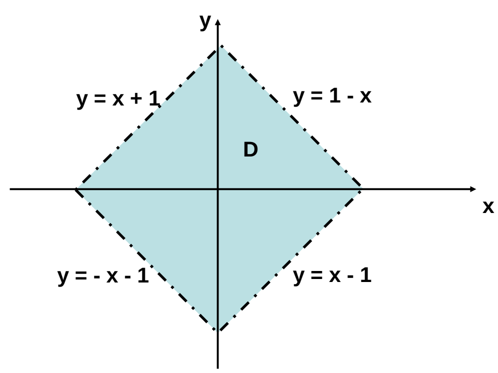



We see that $\partial(C) \cap C \neq \emptyset$, so the set **is not open**. Furthermore, $C \neq \bar{C}$ so the set **is not closed**.

(d) The following functions defined from $\mathbb{R}^2$ into $\mathbb{R}$ are continuous.

$$\begin{aligned} f_1(x,y) &= y - x - 1 \\ f_2(x,y) &= y - 1 + x \\ f_3(x,y) &= y + x + 1 \\ f_4(x,y) &= y - x + 1 \end{aligned}$$

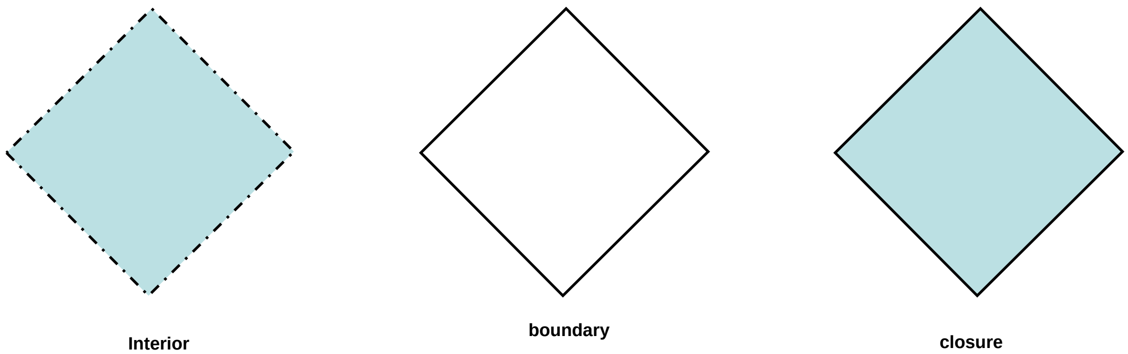
The set $D$




is defined by

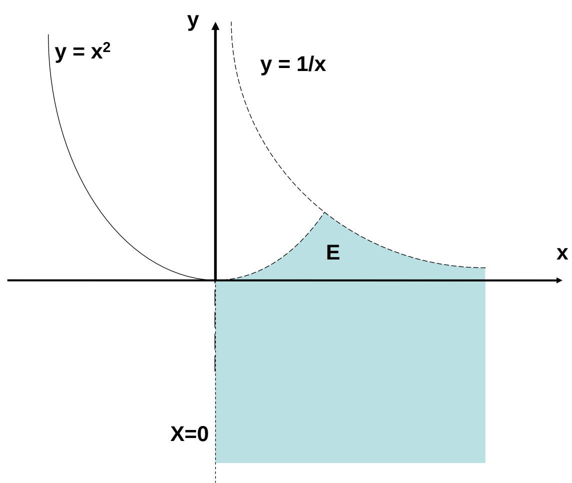
$$D = \{(x,y) \in \mathbb{R}^2 : f_1(x,y) < 0, \quad f_2(x,y) < 0, \quad f_3(x,y) > 0, \quad f_4(x,y) > 0\}$$

so it **is open and convex**. The set $D$ **is bounded** because is contained in the disc of center $(0,0)$ and radius 1.
The interior, boundary and closure of $A$ are represented in the following figure




Since, $\partial(D) \cap D = \emptyset$, the set **is open**.

(e) The graphic representation of $E$ is




The functions

$$\begin{aligned} f_1(x,y) &= y - x^2 \\ f_2(x,y) &= y - 1/x \\ f_3(x,y) &= x \end{aligned}$$

are defined from $\mathbb{R}^2$ into $\mathbb{R}$ and are continuous. The set $E$ is defined by

$$E = \{(x,y) \in \mathbb{R}^2 : f_1(x,y) < 0, f_2(x,y) < 0, f_3(x,y) > 0\}$$

so it **is open**. The set $E$ **is not bounded** because the points

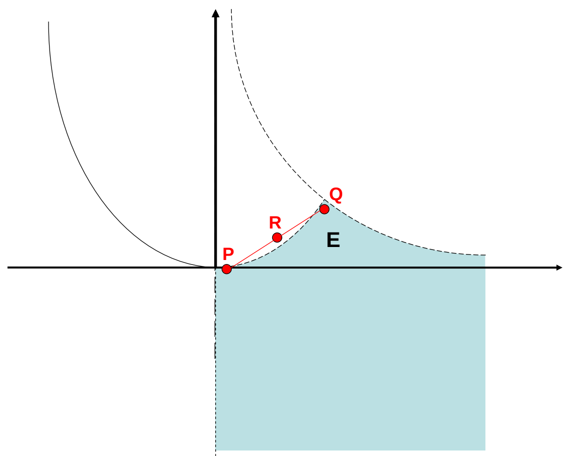
$$(n,0) \quad n = 1,2,\ldots$$

belong to $E$ and

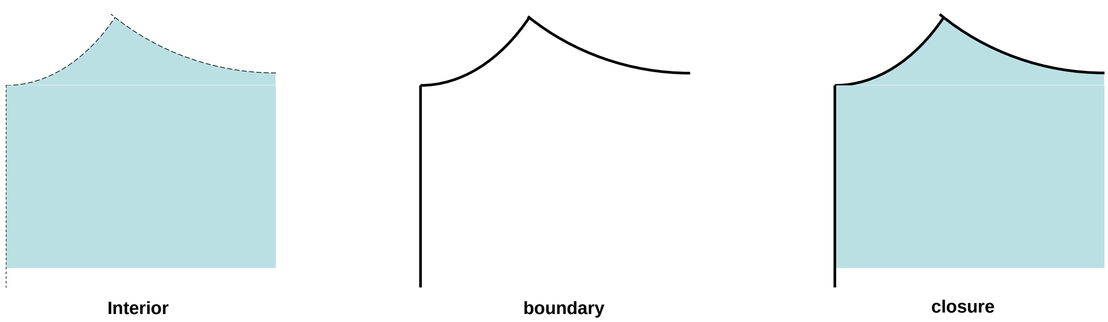
$$\lim_{n\to\infty} \| (n,0) \| = \lim_{n\to\infty} n = +\infty$$

In addition, **it is not convex** because the points $P = (0'2, 0)$ and $Q = (1, 0'8)$ belong to $E$ but the convex combination

$$R = \frac{1}{2}P + \frac{1}{2}Q = (0'6, 0'4)$$

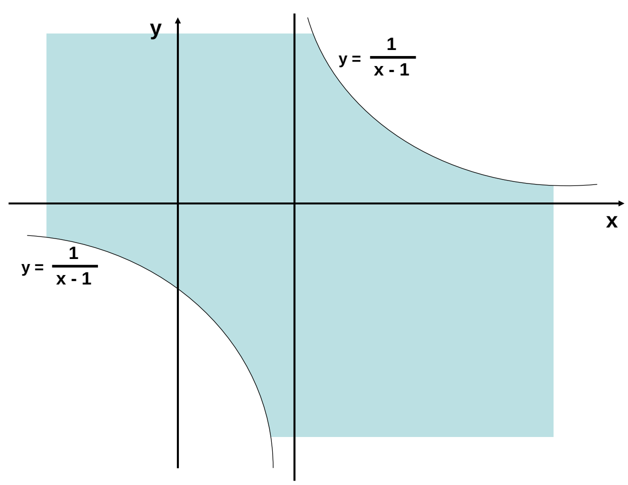



does not belong to $E$, because it does not satisfy the inequality $y < x^2$. The interior, boundary and closure of $E$ are represented in the following figure




Since $\partial(E) \cap E = \emptyset$, the set **is open**.

(f) Graphically, the $F$ is



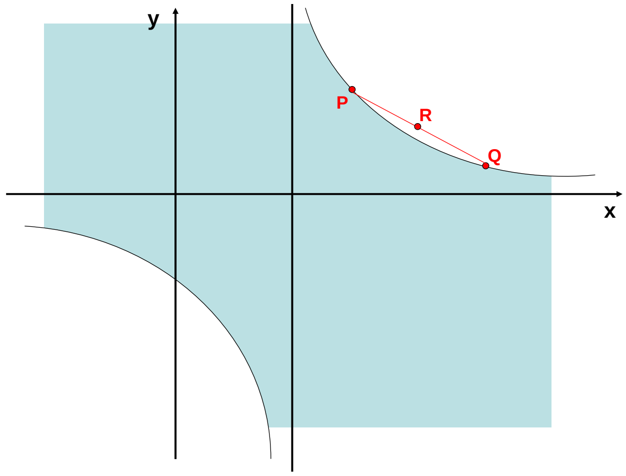

The function $f(x, y) = xy - y$ defined from $\mathbb{R}^2$ into $\mathbb{R}$ is continuous. The set $F$ is $F = \{(x, y) \in \mathbb{R}^2 : f(x, y) \leq 1\}$ so **is closed**. The set $F$ **is not bounded** because the points

$$(n, 0) \quad n = 1, 2, \ldots$$

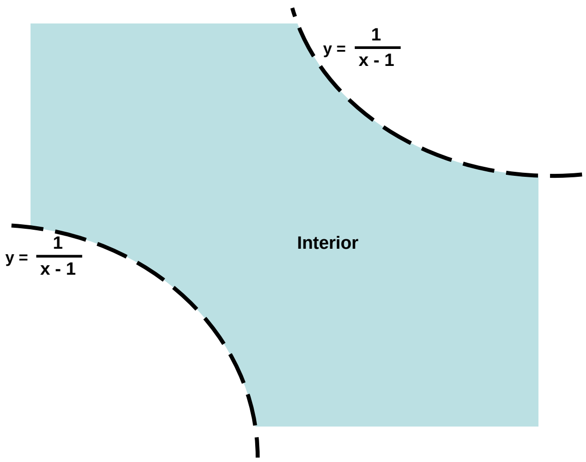
are in $E$ and

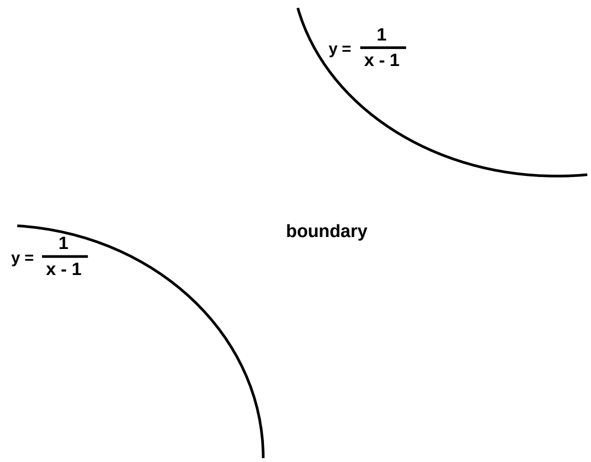
$$\lim_{n\to\infty} \| (n, 0) \| = \lim_{n\to\infty} n = +\infty$$

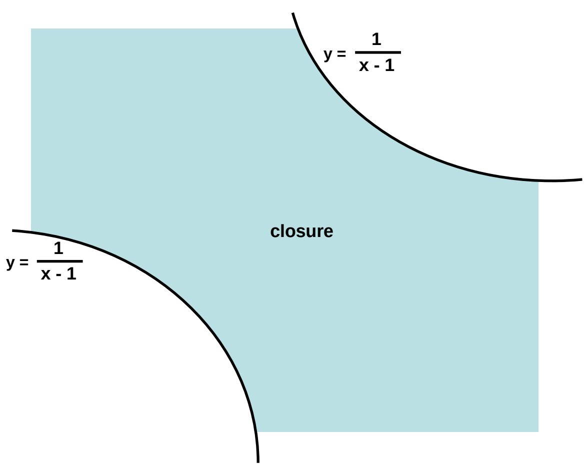
The figure




shows why $F$ **is not convex**. The interior, closure and boundary of $F$ are represented in the following figure

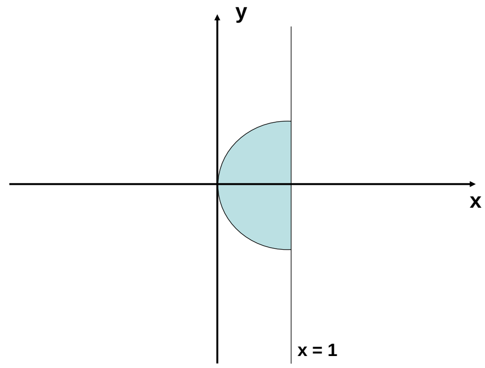


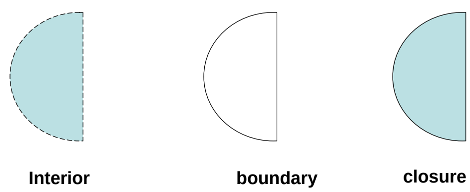




Since $\partial(F) \subset F$, the set $F$ **is closed**.

(g) Graphically the set $G$ is




The functions $f(x,y) = (x-1)^2 + y^2$ and $g(x,y) = x$ defined from $\mathbb{R}^2$ into $\mathbb{R}$ are continuous. The set $G$ is $G = \{(x,y) \in \mathbb{R}^2 : f(x,y) \le 1, \quad g(x,y) \le 1\}$ so it **is closed**. The set $G$ **is bounded** because it is contained in the disc of center $(1,0)$ and radius 1. Further, the set $G$ **is convex**. The interior, boundary and the closure of $G$ are represented in the following figure




Since $\partial(G) \subset G$, the set $G$ is **closed**.

2-2. *Let $A$ be a subset of $\mathbb{R}^2$. Discuss which of the following assertions are true.*

(a) $Int(A) = A - \partial(A)$.
(b) $\partial(A) = \partial(\mathbb{R}^2 - A) = \partial(A^C)$.
(c) $\partial(A)$ *is bounded.*
(d) $A$ *is closed if and only if* $A^C$ *is open.*
(e) $A$ *is bounded if and only if* $A^C$ *is not bounded.*
(f) $A$ *is closed if and only if* $\partial(A) \subset A$.
(g) $A$ *is open if and only if* $(\partial A) \cap A = \emptyset$.

**Solution:**

(a) Yes, because: $x \in \text{Int}(A) \Longleftrightarrow \exists\, \varepsilon > 0 : B(x,\varepsilon) \subset A \Longleftrightarrow \exists\, \varepsilon > 0 : B(x,\varepsilon) \cap (\mathbb{R}^n \setminus A) = \emptyset \Longleftrightarrow x \in A$ and $x \notin \partial A$.
(b) Yes, because: $\partial(\mathbb{R}^n \setminus A) = \overline{\mathbb{R}^n \setminus A} \cap \overline{\mathbb{R}^n \setminus (\mathbb{R}^n \setminus A)} = \overline{\mathbb{R}^n \setminus A} \cap \overline{A} = \partial(A)$.
(c) No. Example: $A = \{(x,y) \in \mathbb{R}^2 : x \ge 0\}$.
(d) Yes. By definition.
(e) No. Example: $A = \{(x,y) \in \mathbb{R}^2 : x \ge 0\}$.
(f) Yes, because: $A$ is closed $\Longleftrightarrow \mathbb{R}^n \setminus A$ is open $\Longleftrightarrow \mathbb{R}^n \setminus A = \text{Int}(\mathbb{R}^n \setminus A)$. But, from (a) and (b), $\text{Int}(\mathbb{R}^n \setminus A) = (\mathbb{R}^n \setminus A) \setminus \partial(\mathbb{R}^n \setminus A) = (\mathbb{R}^n \setminus A) \setminus \partial(A)$. Therefore $A$ is closed $\Longleftrightarrow \mathbb{R}^n \setminus A = (\mathbb{R}^n \setminus A) \setminus \partial A$ $\Longleftrightarrow (\partial A) \subset A$.
(g) Yes, because: $A$ is open $\Longleftrightarrow A = \text{Int}(A) \Longleftrightarrow A = A \setminus \partial A \Longleftrightarrow A \cap A = \emptyset$.