UNIVERSITY CARLOS III OF MADRID

MATHEMATICS FOR ECONOMICS I — FALL 2024

EXERCISES (SOLUTIONS)

CHAPTER 2: Limits and Continuity of Functions of Several Variables.

2-1. *Find the domain of the following functions.*

(a) $f(x,y) = (x^2+y^2-1)^{1/2}$.
(b) $f(x,y) = \dfrac{1}{xy}$.
(c) $f(x,y) = e^x - e^y$.
(d) $f(x,y) = e^{xy}$.
(e) $f(x,y) = \ln(x+y)$.
(f) $f(x,y) = \ln(x^2+y^2)$.
(g) $f(x,y,z) = \sqrt{\frac{x^2+1}{yz}}$.
(h) $f(x,y) = \sqrt{x-2y+1}$.

Solution:

(a) $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \geq 1\}$.
(b) $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$. ($\mathbb{R}^2$ except the axes).
(c) $\mathbb{R}^2$.
(d) $\mathbb{R}^2$.
(e) $\{(x,y) \in \mathbb{R}^2 : x+y > 0\}$.
(f) $\mathbb{R}^2 \setminus \{(0,0)\}$.
(g) $\{(x,y,z) \in \mathbb{R}^3 : yz > 0\}$.
(h) $\{(x,y) \in \mathbb{R}^2 : x-2y \geq -1\}$.

2-2. *Find the range of the following functions.*

(a) $f(x,y) = (x^2+y^2+1)^{1/2}$.
(b) $f(x,y) = \dfrac{xy}{x^2+y^2}$.
(c) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$.
(d) $f(x,y) = \ln(x^2+y^2)$.
(e) $f(x,y) = \ln(1+x^2+y^2)$.
(f) $f(x,y) = \sqrt{x^2+y^2}$.

Solution:

(a) $[1,\infty)$.
(b) **Answer:** $\left[\frac{-1}{2}, \frac{1}{2}\right]$. Let us see why. First, we show the inequality

$$2\ xy| \leq x^2+y^2$$

First, we note that

$$0 \leq (x+y)^2 = x^2+y^2+2xy$$

so,

$$-(x^2+y^2) \leq 2xy$$

On the other hand,

$$0 \leq (x-y)^2 = x^2+y^2-2xy$$

so,

$$2xy \leq x^2+y^2$$

From the above two inequalities, $2\ xy| \leq x^2+y^2$. From here we obtain that

$$2\left|\frac{xy}{x^2+y^2}\right| \leq 1$$

that is,

$$-\frac{1}{2} \leq \frac{xy}{x^2+y^2} \leq \frac{1}{2}$$

Therefore, the image of f is contained in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Finally, we show that it also takes the values $\{-\frac{1}{2}, \frac{1}{2}\}$. Taking $x = y = t$, we have that

$$f(t,t) = \frac{t^2}{t^2 + t^2} = \frac{1}{2}$$

Taking $x = t, y = -t$, we have that

$$f(t,-t) = \frac{-t^2}{t^2 + t^2} = -\frac{1}{2}$$

We conclude that the image of f coincides with the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

(c) **Answer:** $[-1, 1]$. Let us see why. First, we note that

$$\frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2}{x^2 + y^2} - \frac{y^2}{x^2 + y^2}$$

y

$$-1 \leq -\frac{y^2}{x^2 + y^2} \leq \frac{x^2}{x^2 + y^2} - \frac{y^2}{x^2 + y^2} \leq \frac{x^2}{x^2 + y^2} \leq 1$$

Therefore, the image of f is contained in the interval $[-1, 1]$. Finally, we show that it also takes the values $\{-1, 1\}$. Taking $x = 1$, $y = 0$, we have that $f(1, 0) = 1$. Taking $x = 0$, $y = -1$, we have that $f(0, -1) = -1$. We conclude that the image of f coincides with the interval $[-1, 1]$.

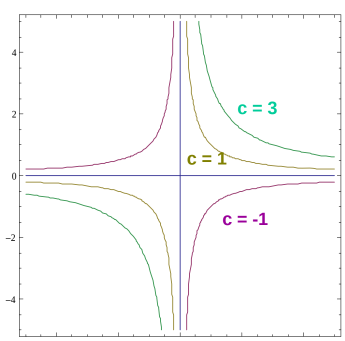
(d) $(-\infty, \infty)$.
(e) $[0, \infty)$.
(f) $[0, \infty)$.

2-3. *Draw the level curves of the following functions.*
(a) $f(x, y) = xy$, $c = 1, -1, 3$.
(b) $f(x, y) = e^{xy}$, $c = 1, -1, 3$.
(c) $f(x, y) = \ln(xy)$, $c = 0, 1, -1$.
(d) $f(x, y) = (x + y)/(x - y)$, $c = 0, 2, -2$.
(e) $f(x, y) = x^2 - y$, $c = 0, 1, -1$.
(f) $f(x, y) = ye^x$, $c = 0, 1, -1$.

Solution:
(a) The level curves are determined by the equation $xy = c$. For $c \neq 0$, this equation is equivalent

$$y = \frac{c}{x}$$

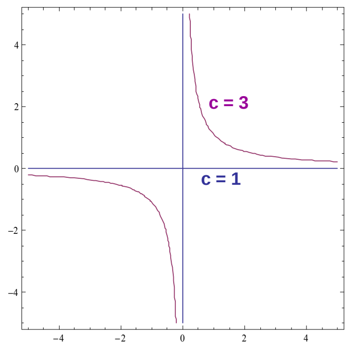
so the level curves are



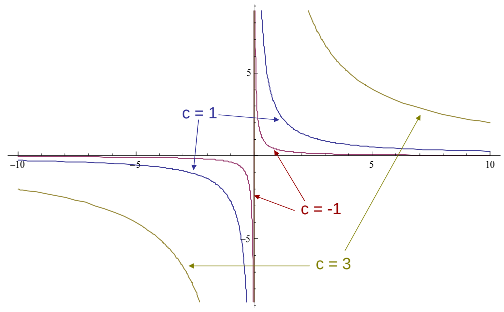
(b) The level curves are determined by the equation $e^{xy} = c$. Therefore, the level curve corresponding to $c \leq 0$ is the empty set. In particular, there is no level curve corresponding to $c = -1$.
For $c > 0$, the level curve satisfies the equation $e^{xy} = c$, so $xy = \ln c$. For $c = 1$, the level curve consists of the points $(x, y) \in \mathbb{R}^2$ such that $xy = 0$. For $c = 3$, the level curve consists of the points $(x, y) \in \mathbb{R}^2$ such that

$$y = \frac{\ln 3}{x}$$

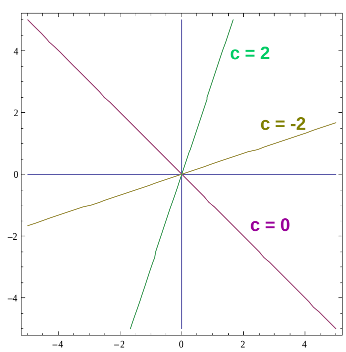
Graphically,



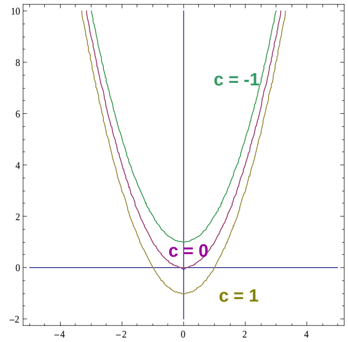
(c) The level curves are determined by the equation $\log(xy) = c$ which is the same as $xy = e^c$. Graphically,



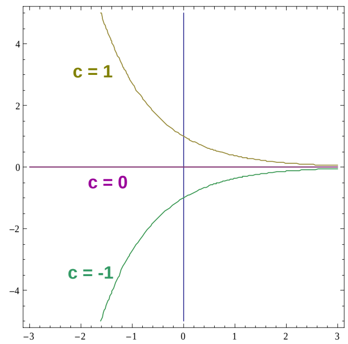
(d) The level curves are determined by the equation $x+y = c(x-y)$, which is the same as $(1+c)y = (c-1)x$. Graphically,



(e) The level curves satisfy the equation $y = x^2 - c$. Graphically,



(f) The level curves satisfy the equation $y = ce^{-x}$. Graphically,



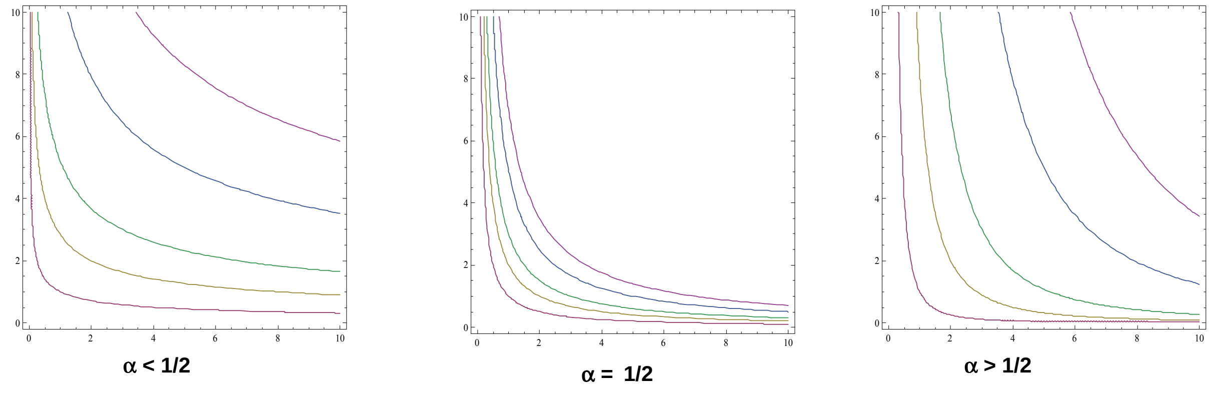
2-4. *Let $f(x, y) = Cx^{\alpha}y^{1-\alpha}$, with $0 < \alpha < 1$ and $C > 0$ be the Cobb-Douglas production function, where x (resp. y) represents units of labor (resp. capital) and f are the units produced.*

(a) *Represent the level curves of f.*

(b) *Show that if one duplicates labor and capital then, production is doubled, as well.*

Solution:

(a) The level curves are,



(b) $f(x, y) = Cx^{\alpha}y^{1-\alpha}$, $f(2x, 2y) = C\,(2x)^{\alpha}\,(2y)^{1-\alpha} = 2Cx^{\alpha}y^{1-\alpha} = 2f(x, y)$.

2-5. *Study the existence and the value of the following limits.*

(a) $\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$.

(b) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$.

(c) $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^4+y^2}$.
(d) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+2y^2}$.
(e) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.
(f) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$.
(g) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2}$.

Solution:

(a) $\left[(\frac{x}{x^2+y^2})\right]_{y=kx} = \frac{x}{x^2+k^2x^2} = \frac{1}{x(1+k^2)}$ and $\lim_{x\to 0}\left(\frac{1}{x(1+k^2)}\right)$ does not exist. The limit does not exist.

(b) We show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Note

$$0 \leq |f(x,y)| = \left|\frac{xy^2}{x^2+y^2}\right| \leq \frac{|x|(x^2+y^2)}{x^2+y^2} = |x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$$

The function

$$g(x,y) = \sqrt{x^2+y^2}$$

is continuous. Therefore,

$$\lim_{(x,y)\to(0,0)} g(x,y) = g(0,0) = 0$$

and we conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

(c) On the one hand,

$$\left[(\frac{3x^2y}{x^4+y^2})\right]_{y=kx} = 3x^3\frac{k}{x^4+k^2x^2} = 3x\frac{k}{x^2+k^2}$$

so

$$\lim_{x\to 0}\left[(\frac{3x^2y}{x^4+y^2})\right]_{y=kx} = 0$$

On the other hand,

$$\left[(\frac{3x^2y}{x^4+y^2})\right]_{y=x^2} = \frac{3}{2}$$

Therefore, the limit does not exist.

(d) $\left[(\frac{x^2-y^2}{x^2+2y^2})\right]_{y=kx} = \frac{x^2-k^2x^2}{x^2+2k^2x^2} = \frac{1-k^2}{1+2k^2}$, depends on k. Therefore, the limit does not exist.

(e) $\left[(\frac{xy}{x^2+y^2})\right]_{y=kx} = x^2\frac{k}{x^2+k^2x^2} = \frac{k}{1+k^2}$, depends on k

(f) We show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Let $\varepsilon > 0$. Take $\delta = \varepsilon$ and suppose that $0 < \|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$. Then,

$$|f(x,y)-0| = \left|\frac{x^2y}{x^2+y^2}\right| \leq \frac{(x^2+y^2)|y|}{x^2+y^2} = |y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} = \delta = \varepsilon.$$

(g) We show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Let $\varepsilon > 0$. Take $\delta = \sqrt{\varepsilon}$ and suppose that $0 < \|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$. Then,

$$|f(x,y)-0| = \left|\frac{xy^3}{x^2+y^2}\right| = \left|\frac{y^2}{x^2+y^2}xy\right| \leq \frac{x^2+y^2}{x^2+y^2}|xy| = |xy| = |x||y| = \sqrt{x^2}\sqrt{y^2}$$
$$\leq \sqrt{x^2+y^2}\sqrt{x^2+y^2} = x^2+y^2 = \delta^2 = \varepsilon$$

And the limit is 0

2-6. *Study the continuity of the following functions.*

(a) $f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

(b) $f(x,y) = \begin{cases} \frac{xy+1}{y}x^2 & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$.

(c) $f(x,y) = \begin{cases} \frac{x^4y}{x^6+y^3} & \text{if } y \neq -x^2 \\ 0 & \text{if } y = -x^2 \end{cases}$.

(d) $f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2} & si\ (x,y) \neq (0,0) \\ 0 & si\ (x,y) = (0,0) \end{cases}$.

Solution:

(a) The function $\frac{x^2y}{x^3+y^3}$ is not continuous at the points $\{(x,y) : x = -y\}$. (The limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$$

does not exist. This can be shown by taking curves of the form $y = kx$.)

(b) The function $\frac{xy+1}{y}x^2$,

(i) is continuous at the points (x,y) such that $y \neq 0$.

(ii) is not continuous at the points of the form $(x_0, 0)$ with $x_0 \neq 0$. Since, the limit

$$\lim_{y\to 0}(f(x_0,y)) = \lim_{y\to 0} x_0^3 + \frac{x_0^2}{y}$$

does not exist if $x_0 \neq 0$.

(iii) It is not continuous at $(0,0)$ because

$$\lim_{x\to 0} f(x, kx^2) = \lim_{x\to 0}\left(x^3 + \frac{1}{k}\right) = \frac{1}{k}$$

which depends on k.

(c) The function

$$\frac{x^4y}{x^6+y^3}$$

is continuous at the points (x,y) such that $y \neq -x^2$. On the other hand, at the points of the form $(a, -a^2)$ is not continuous because,

(i) If $a \neq 0$, we have that

$$\lim_{y\to -a^2} f(a,y)$$

does not exist because the numerator approaches $-a^6 \neq 0$ whereas the denominator approaches 0.

(ii) The limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist because

$$\lim_{t\to 0} f(t,t^2) = \lim_{t\to 0} \frac{t^6}{t^6+t^6} = \frac{1}{2}$$

whereas the value of the iterated limits is 0.

(d) The function $\frac{xy^3}{x^2+y^2}$ is a quotient of polynomials and the denominator only vanishes at the point $(x,y) = (0,0)$. Hence, the function is continuous at every point $(x,y) \neq (0,0)$.
At the point $(0,0)$ the function is also continuous because we have already proved in another problem that

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2} = 0$$

We conclude that the function is continuous in all of $\mathbb{R}^2$.

2-7. *Consider the set* $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x, \quad \leq 1, \quad 0 \leq y \leq 1\}$ *and the function* $f\colon A \longrightarrow \mathbb{R}^2$*, defined by*

$$f(x,y) = \left(\frac{x+1}{y+2}, \frac{y+1}{x+2}\right)$$

Are the hypotheses of Brouwer's Theorem satisfied? Is it possible to determine the fixed point(s)?

Solution: Brouwer's Theorem: Let A be a compact, non-empty and convex subset of R^n and let $f : A \to A$ be a continuous function. Then, f has a unique fixed point. (That is, a point $a \in A$, such that $f(a) = a$). The set A is not empty, compact and convex. The function f is continuous if $y \neq -2$ and $x \neq -2$. Therefore, f is continuous on A and Brouwer's Theorem applies.

If (x, y) is the fixed point of f, then

$$x = \frac{x+1}{y+2}$$
$$y = \frac{y+1}{x+2}$$

that is,

$$xy = 1 - x$$
$$xy = 1 - y$$

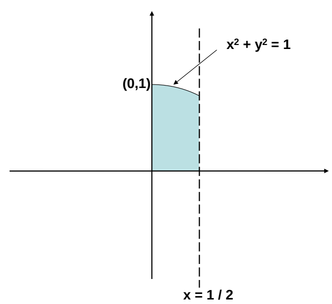
Therefore $x = y$ satisfies the equation $x^2 + x - 1 = 0$ whose solutions are

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

The only solution in the set A is $(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2})$.

2-8. *Consider the function $f(x, y) = 3y - x^2$ defined on the set $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, \quad 0 \leq x < 1/2, \quad y \geq 0\}$. Draw the set D and the level curves of f. Does f have a maximum and a minimum on D?*

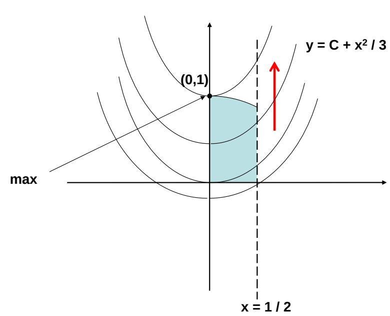
Solution: The set D is the following



Note that D is not compact (since it is not closed). It does not contain the point $(1/2, 0)$. On the other hand, the level curves of f are of the form

$$y = C + \frac{x^2}{3}$$

Graphically, (the red arrow points in the direction of growth)



We see that f attains a maximum at the point $(0, 1)$, but attains no minimum on A.

2-9. *Consider the sets $A = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 | -1 \leq x \leq 1, -1 \leq y \leq 1\}$ and the function*

$$f(x, y) = \frac{(x+1)\left(y + \frac{1}{5}\right)}{y + \dfrac{1}{2}}$$

What can you say about the extreme points of f on A and B?

Solution: The function

$$f(x, y) = \frac{(x+1)\left(y + \frac{1}{5}\right)}{y + \frac{1}{2}}$$

is continuous if $y \neq -1/2$ and so, is continuous in the set A, which is compact. By Weierstrass' Theorem, f attains a maximum and a minimum on A.

But, for example, the point $(0, -1/2) \in \text{Int}B$ and

$$\lim_{y \to \left(\frac{-1}{2}\right)^+} f(0, y) = -\infty, \qquad \lim_{y \to \left(\frac{-1}{2}\right)^-} f(0, y) = +\infty$$

so f does not attain neither a maximum nor a minimum on B.

2-10. *Consider the set*

$$A = \{(x, y) \in \mathbb{R}^2 \colon 0 \leq y \leq \ln x, 1 \leq x \leq 2\}.$$

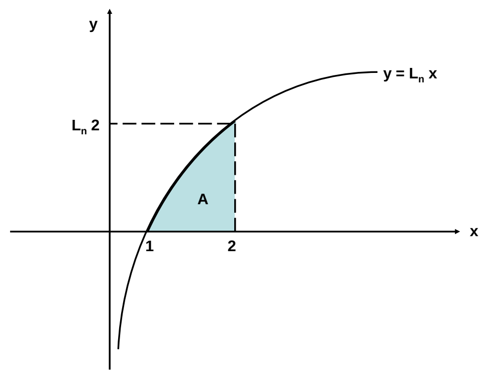
(a) *Draw the set A, its boundary and its interior. Discuss whether the set A is open, closed, bounded, compact and/or convex. You must explain your answer.*

(b) *Prove that the function $f(x, y) = y^2 + (x-1)^2$ has a maximum and a minimum on A.*

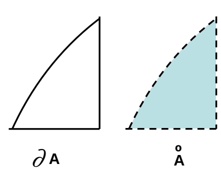
(c) *Using the level curves of $f(x, y)$, find the maximum and the minimum of f on A.*

Solution:

(a) The set A is



The boundary and the interior are



Since $\partial A \subset A$, the set A is closed. It is not open because $\partial A \cap A \neq \emptyset$. Another way of proving this, would be to consider the sets $A_1 = \{(x, y) \in \mathbb{R}^2 \colon 0 \leq y\}$, $A_2 = \{(x, y) \in \mathbb{R}^2 \colon 1 \leq x \leq 2\}$. The set $A_3 = \{(x, y) \in \mathbb{R}^2 \colon y \leq \log(x)\}$ is also closed since the function $g(x, y) = log(x) - y$ is continuous. Therefore, $A = A_1 \cap A_2 \cap A_3$ is a closed set.

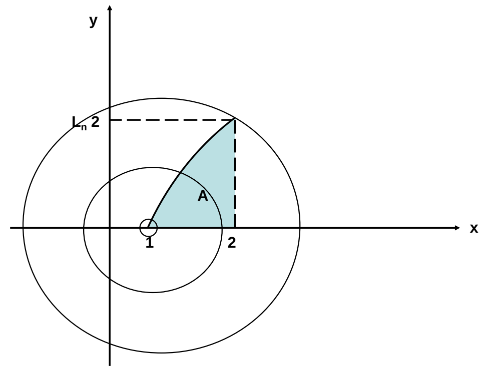
The set A is bounded since $A \subseteq B(0, r)$ with $r > 0$ large enough. Since it is closed and bounded the set A is compact. The set A is convex since is the region under the graph of $f(x) = \ln x$ in the interval $[1, 2]$ and the function $\ln x$ is concave.

(b) The function f is continuous in $\mathbb{R}^2$, since it is a polynomial. In particular, the function is continuous in the set A. Furthermore, the set A is compact. By Weierstrass' Theorem, the function attains a maximum and a minimum on A.

(c) The equations defining the level curves of f are

$$f(x, y) = y^2 + (x-1)^2 = C$$

These sets are circles centered at the point $(1, 0)$ and radius $\sqrt{C}$, for $C \geq 0$.



Graphically, we see that the maximum is $f(2, \ln 2) = 1 + (\ln 2)^2$ and is attained at the point $(2, \ln 2)$. The minimum is $f(1, 0) = 0$ and is attained at the point $(1, 0)$.

2-11. *Consider the set* $A = \{(x, y) \in \mathbb{R}^2 : x, y > 0; \ln(xy) \geq 0\}$.

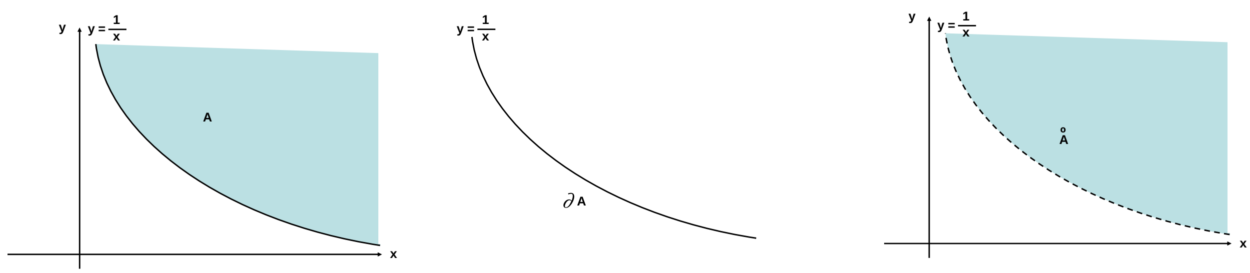
(a) *Draw the set* A*, its boundary and its interior. Discuss whether the set* A *is open, closed, bounded, compact and/or convex. You must explain your answer.*

(b) *Consider the function* $f(x, y) = x + 2y$*. Is it possible to use Weierstrass' Theorem to determine whether the function attains a maximum and a minimum on* A*? Draw the level curves of* f*, indicating the direction in which the function grows.*

(c) *Using the level curves of* f*, find graphically (i.e. without using the first order conditions) if* f *attains a maximum and/or a minimum on* A*.*

Solution:

(a) The equation $\ln(xy) \geq 0$ is equivalent to $xy \geq 1$. Since $x, y > 0$, the set is $A = \{(x, y) \in \mathbb{R}^2 : y \geq 1/x, \quad x > 0\}$. Graphically,



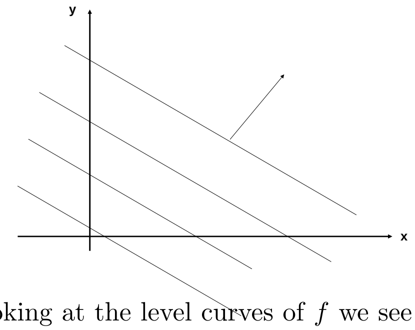
The boundary is the set $A = \{(x, y) \in \mathbb{R}^2 : y = 1/x, \quad x > 0\}$. The interior is the set $\overset{\circ}{A} = \{(x, y) \in \mathbb{R}^2 : y > 1/x, \quad x > 0\}$.

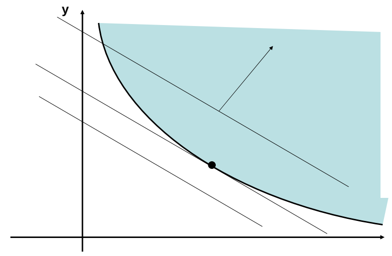
Since $\partial A \cap A \neq \emptyset$, the set A is not open. Furthermore, $\partial A \subset A$ so the set A is closed. Graphically, we see that A is not bounded. The set A is not compact (since is not bounded). We may show that the set A is convex in two different ways.

(i) Consider the function $g(x) = \frac{1}{x}$. It is easy to show that the function is convex. Therefore, the set $\{(x, y) \in \mathbb{R}^2 \in \mathbb{R} : x > 0, y \geq \frac{1}{x}\}$ is also convex.

(ii) Consider the function $g(x, y) = \ln(xy) = \ln x + \ln y$, defined on the convex set $D = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}$. The Hessian matrix of this function is $\mathrm{H}\, g = \begin{pmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}$, which is negative definite. From here we conclude that the function g is concave in D. Since, $A = \{(x, y) \in D : g(x, y) \geq 0\}$, the set A is convex.

(b) We may no apply Weierstrass' Theorem since the set A is not compact. The level curves of $f(x, y) = x + 2y$ are sets of the form $\{(x, y) \in \mathbb{R}^2 : y = C - x/2\}$ which are straigt lines. Graphically (the vector indicates the direction of growth)





(c) Looking at the level curves of f we see that the funtion does not attain a (local o global) maximum on A. The global minimum is attained at the point of tangency of the straight line $y = C - x/2$ with the graph of $y = 1/x$,
This point satisfies that

$$-\frac{1}{2} = -\frac{1}{x^2}$$

that is $x = \pm\sqrt{2}$. And since $x > 0$, the minimum is attained at the point $\left(\sqrt{2}, 1/\sqrt{2}\right)$,