Session 9
Mathematics for Economics I

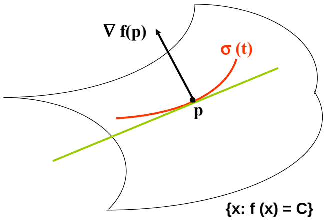
Chapter 3: Differentiability. Part IV: Line and tangent planes. Taylor's polynomial of order 1

Degrees in Economics, International-Studies-and-Economics and Law-and-Economics

Universidad Carlos III de Madrid

Gradient and level curves.

- Consider the level surface $S_C = \{x \in D : f(x) = C\}$.
- Let $\sigma : \mathbb{R} \to \mathbb{R}^n$ be a differentiable curve and suppose that $\sigma(t) \in S_C$ for all $t \in \mathbb{R}$. That is $f(\sigma(t)) = c$ for every $t \in \mathbb{R}$.
- Then, $0 = \frac{d}{dt} f(\sigma(t)) = \nabla f(\sigma(t)) \cdot \frac{d\sigma}{dt}$
- That is $\nabla f(\sigma(t))$ and $d\sigma(t)/dt$ are perpendicular for every $t \in \mathbb{R}$.



- $\nabla f(\sigma(t))$ is perpendicular to the surface $S_C = \{x \in D : f(x) = C\}$.

Example.

- Consider the surface given by the equation $3x^2 + 2y^2 + 5z^2 = 56$.
- The gradient of the function $f(x, y, z) = 3x^2 + 2y^2 + 5z^2$ is $\nabla f(x, y, z) = (6x, 4y, 10z)$.
- At the point $p = (-1, 2, -3)$ we get $\nabla f(-1, 2, -3) = (-6, 8, -30)$.
- The equation of the tangent plane is $-6(1 + x) + 8(-2 + y) - 30(3 + z) = 0$ or $-6x + 8y - 30z = 112$.
- The parametric equations of the normal line are $(x, y, z) = (-1, 2, -3) + t(-6, 8, -30)$.
- That is, $x = -1 - 6t$, $y = 2 + 8t$, $z = -3 - 30t$.

Plane tangent to the graph of a function.

- The graph of f is the set $G = \{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2\}$.
- Define $g(x, y, z) = f(x, y) - z$. The graph of f may be written as $G = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$.
- An equation for the tangent plane to G at $p = (a, b)$ is

$$\nabla g(a, b, f(a, b)) \cdot ((x, y, z) - (a, b, f(a, b))) = 0$$

- Since, $\nabla g(a, b, f(a, b)) = \left(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1 \right)$.
- We obtain

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b)$$

Taylor polynomial of first order.

- Let $f \in C^1(D)$, $p \in D$. The Taylor polynomial of first order at p is

$$P_1(x) = f(p) + \nabla f(p) \cdot (x - p)$$

- If $f(x, y)$ is a function of two variables and $p = (a, b)$, then Taylor's first order polynomial for the function f around the point $p = (a, b)$ is the polynomial

$$P_1(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b)$$

Taylor's first order polynomial.

- The function f is differentiable at (a, b) if

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - P_1(x,y)|}{\|(x-a, y-b)\|} = 0$$

- That is, if the tangent plane is a 'good' approximation to the value of the function

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)$$

Example.

- $f(x, y) = -2y + xy^3 - 2xy + 4x - y^2 + 1$ and $p = (-1, 1)$. Let us compute the equation of the tangent plane to the graph of the function f at the point $(p, f(p))$.
- The equation of the tangent plane is

$$\begin{aligned} z = f(-1,1) + \nabla f(p) \cdot (x+1, y-1) = \\ = -5 + (3, -5) \cdot (x+1, y-1) = \\ = -5 + 3(x+1) - 5(y-1) \end{aligned}$$

- It coincides with Taylor's first order polynomial of f at p.

Example.

- Consider the function $f(x, y) = 2x^2y - xy + 2x - 2y^2 - 15y + 1$ and the point $p = (1, 2)$.
- We have $\nabla f(x, y) = \big(4xy - y + 2, 2x^2 - x - 4y - 15\big)$.
- $\nabla f(1, 2) = (8, -22)$.
- Thus, the tangent plane to the graph of the function f at the point $(p, f(p))$ is

$$\begin{aligned} z &= f(1, 2) + \nabla f(p) \cdot (x - 1, y - 2) \\ &= -33 + (8, -22) \cdot (x - 1, y - 2) = \\ &= -33 + 8(-1 + x) - 22(-2 + y) \end{aligned}$$