Session 8
Mathematics for Economics II

Chapter 3. Differentiation. Part III: The chain rule.

Degrees in Economics, International-Studies-and-Economics and Law-and-Economics

Universidad Carlos III de Madrid

The Jacobian.

- The Jacobian matrix of f at p is the $m \times n$ matrix

$$\mathrm{D}\, f(p) = \begin{pmatrix} \frac{\partial f_1(p)}{\partial x_1} & \frac{\partial f_1(p)}{\partial x_2} & \dots & \frac{\partial f_1(p)}{\partial x_n} \\ \frac{\partial f_2(p)}{\partial x_1} & \frac{\partial f_2(p)}{\partial x_2} & \dots & \frac{\partial f_2(p)}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m(p)}{\partial x_1} & \frac{\partial f_m(p)}{\partial x_2} & \dots & \frac{\partial f_m(p)}{\partial x_n} \end{pmatrix}.$$

- Examples.
- For $f : D \subset \mathbb{R}^n \to \mathbb{R}$, what is the difference between $\mathrm{D}\, f(p)$ and $\nabla f(p)$?

The chain rule.

Theorem (The chain rule)

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ *and* $f : \mathbb{R}^m \to \mathbb{R}^l$. *Suppose that* g *is differentiable at* $p \in \mathbb{R}^n$ *and that* f *is differentiable at* $g(p) \in \mathbb{R}^m$. *Then, the function* $f \circ g$ *is differentiable at* p *and*

$$\mathrm{D}(f \circ g)(p) = \mathrm{D}\, f(g(p))\, \mathrm{D}\, g(p)$$

- The expression $\mathrm{D}(f \circ g)(p) = \mathrm{D}\, f(g(p))\, \mathrm{D}\, g(p)$ consists of the product of 2 matrices.

Example of Chain Rule.

- Let $\sigma : \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (x(t), y(t))$, $f : \mathbb{R}^2 \to \mathbb{R}$. Then, the chain rule says

$$\begin{aligned}
\frac{d}{dt} f(x(t), y(t)) &= \mathrm{D}(f \circ \sigma)(t) = \mathrm{D}\, f(x,y)\Big|_{\substack{x=x(t)\\ y=y(t)}} \mathrm{D}\,\sigma(t) \\
&= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}\Big|_{\substack{x=x(t)\\ y=y(t)}} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \\
&= \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t)
\end{aligned}$$

- In general, if $\sigma(t) = (\sigma_1(t), \sigma_2(t), \ldots, \sigma_n(t))$, we have $\frac{d}{dt} f(\sigma(t)) = \nabla f(\sigma(t)) \cdot \frac{d\sigma}{dt}$
- **Example:** $f(x, y = xy + y^2$, $x(t) = e^t$, $y(t) = 2t + 1$.Then, $\frac{d}{dt} f(x(t), y(t)) = y(t)x'(t) + (x(t) + 2y(t))y'(t) =$ $(2t + 1)e^t + 2(e^t + 2t + 1)$.

Special case of the chain rule.

- $g(s,t) = (x(s,t), y(s,t)) : \mathbb{R}^2 \to \mathbb{R}^2$ and $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$. Then, $(f \circ g)(s,t) = f(g(s,t)) = f\,(x(s,t), y(s,t))$. The chain rule says that

$$\begin{aligned}
\frac{\partial (f \circ g)}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\
\frac{\partial (f \circ g)}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}
\end{aligned}$$

Special case of the chain rule.

- $g(s,t) = (x(s,t), y(s,t), z(s,t)) : \mathbb{R}^2 \to \mathbb{R}^3$ and $f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$. Then, $(f \circ g)(s,t) = f(g(s,t)) = f\left(x(s,t), y(s,t), z(s,t)\right)$. The chain rule says that

$$\begin{aligned} \frac{\partial(f \circ g)}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \\ \frac{\partial(f \circ g)}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \end{aligned}$$

Example.

- Le $f(x,y,z):\mathbb{R}^3 \longrightarrow \mathbb{R}$ and $x(u,v), y(u,v), z(u,v):\mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y,z) = x^2y + xz$$

$$x(u,v) = e^u, \quad y(u,v) = uv, \quad z(u,v) = \ln v$$

- Consider the composition $h:\mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $h(u,v) = f(x(u,v), y(u,v), z(u,v))$.
- We are going to use the the chain rule to compute

$$\frac{\partial h}{\partial u}(0,1), \quad \frac{\partial h}{\partial v}(0,1)$$

Example.

- First, $x(0,1) = 1$, $y(0,1) = 0$, $z(0,1) = 0$.
- We have, $Df(x,y,z) = \big(2xy + z, x^2, x\big)$.
- So, $Df(1,0,0) = (0,1,1)$.
- Write $g(u,v) = (x(u,v), y(u,v), z(u,v)) = (uv, u-v, u+2v)$.
- Then,

$$Dg(u,v) = \begin{pmatrix} e^u & 0 \\ v & u \\ 0 & \frac{1}{v} \end{pmatrix}, \quad Dg(0,1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example.

- By the chain rule,

$$\begin{pmatrix} \frac{\partial h}{\partial u}(0,1) & \frac{\partial h}{\partial v}(0,1) \end{pmatrix} = Df(1,0,0)\, Dg(0,1) =$$

$$= (0,1,1) \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = (1,1)$$

- Therefore,

$$\frac{\partial h}{\partial u}(0,1) = 1 \quad \frac{\partial h}{\partial v}(0,1) = 1$$

- Repeat the computations using

$$\begin{aligned} \frac{\partial (f \circ g)}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \\ \frac{\partial (f \circ g)}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} \end{aligned}$$

Ejemplo.

- Consider the funcions

$$f(u,v) = uv^2$$

and

$$u(x,y,z) = x + yz, \quad v(x,y,z) = yz - x$$

- We have,

$$\begin{aligned}
\frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\
\frac{\partial h}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \\
\frac{\partial h}{\partial z} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z}
\end{aligned}$$

Ejemplo.

- That is,

$$\begin{aligned}
\frac{\partial h}{\partial x} &= v^2 + 2uv \times (-1) = 3x^2 - 2xyz - y^2z^2 \\
\frac{\partial h}{\partial y} &= v^2 z + 2uvz = -x^2 z - 2xyz^2 + 3y^2z^3 \\
\frac{\partial h}{\partial z} &= v^2 y + 2uvy = -x^2 y - 2xy^2 z + 3y^3z^2
\end{aligned}$$

Example.

- Consider the Cobb-Douglas production function

$$f(K,L) = 5K^{1/3}L^{2/3}$$

- where f are the units produced, K is capital and L is labor.
- Suppose that capital and labor change with time

$$K = K(t), \quad L = L(t)$$

  Then the production function

$$f(K(t), L(t))$$

  is also a function of time.
- What is the rate of change of the production at a given time?

Example.

- We may answer this question using the chain rule.

$$\begin{aligned}\frac{df(K(t),L(t))}{dt} &= \frac{\partial f}{\partial K}\frac{dK}{dt}+\frac{\partial f}{\partial L}\frac{dL}{dt}\\ &= \frac{5}{3}K^{-2/3}L^{2/3}\frac{dK}{dt}+\frac{10}{3}K^{1/3}L^{-1/3}\frac{dL}{dt}\end{aligned}$$

Example.

- Suppose an agent has the following differentiable utility function

$$u(x, y)$$

- where x is a consumption good and y is air pollution. Then, the utility of the agent is increasing in x and decreasing in y,

$$\begin{aligned} \frac{\partial u}{\partial x} &> 0 \\ \frac{\partial u}{\partial y} &< 0 \end{aligned}$$

- Suppose that the production of x units of the good generates $y = f(x)$ units of pollution,
- What is the optimal level of consumption of x?

Example.

- The utility of the agent when he consumes x units of the good and $y = f(x)$ units of pollution are generated is

$$u(x, f(x))$$

- The agents maximizes this utility function.
- The first order condition is

$$\frac{du(x, f(x))}{dx} = 0$$

- Using the chain rule we obtain that the equation

$$0 = \frac{\partial u}{\partial x}(x, f(x)) + \frac{\partial u}{\partial y}(x, f(x))f'(x)$$

determines the optimal level of production of the good.