Session 7
Mathematics for Economics I

Chapter 3: Partial Derivatives and Differentiation. Part II

Degrees in Economics, International-Studies-and-Economics and Law-and-Economics

Universidad Carlos III de Madrid

Directional derivatives.

- $D_v f(p) = \lim_{t \to 0} \frac{f(p+tv)-f(p)}{t}$ is the derivative of f at p along (the vector) v.
- If $\|v\| = 1$, then $D_v f(p)$ is the directional derivative of f at p in the direction of (the vector) v.
- $f(x, y) = xy$, $p = (1, -1)$, $v = (3, 4)$. Then, $p + tv = (1 + 3t, -1 + 4t)$ and

$$\begin{aligned} D_v f(p) &= \lim_{t \to 0} \frac{f(1+3t, -1+4t) - f(1, -1)}{t} \\ &= \lim_{t \to 0} \frac{(1+3t)(-1+4t)+1}{t} = 1 \end{aligned}$$

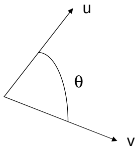
- And, since $\|v\| = \sqrt{3^2 + 4^2} = 5$, the directional derivative of f at p in the direction of v is $\frac{1}{\|v\|} D_v f(p) = \frac{1}{5}$.

Directional derivatives and the gradient.

- If $f : D \to \mathbb{R}$ is differentiable at $p \in D$, then $D_v f(p) = \nabla f(p) \cdot v$.
- Let $f(x, y) = xy$, $p = (1, -1)$, $v = (3, 4)$. Then,
  $\nabla f(p) = (y, x)\Big|_{\substack{x=1 \\ y=-1}} = (-1, 1)$ and
  $D_v f(p) = \nabla f(p) \cdot v = (-1, 1) \cdot (3, 4) = -3 + 4 = 1.$

Interpretation of the gradient.

- $D_v f(p) = \nabla f(p) \cdot v = \|\nabla f(p)\| \, \|v\| \cos\theta$, where



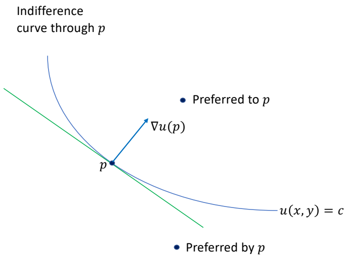
- Thus, $D_v f(p)$
  - attains a maximum when $\theta = 0$, that is, when the vectors $\nabla f(p)$ and v point in the same direction.
  - attains a minimum when $\theta = \pi$, that is, when the vectors $\nabla f(p)$ and v point in the opposite directions.
  - is zero when $\theta = \pi/2$ or $\theta = 3\pi/2$, that is, when the vectors $\nabla f(p)$ and v are perpendicular.

Interpretation of the gradient.

- The function f grows the fastest in the direction of $\nabla f(p)$.
- The function f decreases the fastest in the direction opposite to $\nabla f(p)$.
- Suppose $c = f(p)$. $\nabla f(p)$ is perpendicular to the level surface/curve given by $f(x) = c$. The function f remains constant if we move along a curve that is perpendicular to $\nabla f(p)$.

Example.

- Consider a consumer with utility function u. Then,
  1. The gradient $\nabla u(p)$ points towards the bundles of goods which are preferred to p.
  2. The gradient $\nabla u(p)$ is perpendicular to the line tangent to the indifference curve that goes through p.



Example.

- Consider the function $f(x, y, z) = 8x^2 + 4xz + 8x + y^4 + 32y + z^2 + 16$ defined in $\mathbb{R}^3$.
- We compute $D_v f(-1, 0, 1)$ for v=(2, 1, -1).
- We have

$$\nabla f(x, y, z) = \left(16x + 4z + 8, 4y^3 + 32, 4x + 2z\right)$$

- Since, $\nabla f(-1, 0, 1) = (-4, 32, -2)$,
- we obtain $D_v f(-1, 0, 1) = (-4, 32, -2) \cdot (2, 1, -1) = 26$.

Example.

- Consider the function $f(x, y, z) = ye^{xz} - x^2 - yz^2$, defined in $\mathbb{R}^3$.
- We compute the gradient of f at the point $p = (0, 1, 1)$ and determine for what values of a, b, c we have that $D_v f(p) = 0$.
- We have

$$\nabla f(x, y, z) = (yze^{xz} - 2x, e^{xz} - z^2, xye^{xz} - 2yz)$$

- So,

$$\nabla f(0, 1, 1) = (1, 0, -2)$$

- Therefore

$$D_v f(p) = \nabla(p) \cdot v = (1, 0, -2) \cdot (a, b, c) = a - 2c$$

- And $D_v f(p) = 0$ iff $a = 2c$ with $b, c \in \mathbb{R}$ arbitrary.

Example.

- Consider the function $f(x, y) = x^2 + y^2$, defined in $\mathbb{R}^2$ and the level curve S given by $x^2 + y^2 = 5$. Let us compute the tangent line at the point $p = (1, 2) \in S$.
- $\nabla f(x, y) = (2x, 2y)$, $\nabla f(x, y) = (2, 4)$
- The equation of the tangent line to the curve $x^2 + y^2 = 5$ at the point $p = (1, 2) \in S$ is given by

$$(2, 4) \cdot (x - 2, y - 4) = 0$$