Session 6
Mathematics for Economics I

Chapter 3: Partial Derivatives and Differentiation. Part I

Degrees in Economics, International-Studies-and-Economics and Law-and-Economics

Universidad Carlos III de Madrid

Partial Derivatives.

- In all this chapter, D will denote an open subset of $\mathbb{R}^n$ and we let $p \in D$.
- The partial derivative of $f : D \to \mathbb{R}$ with respect to the i-th variable at the point p is the limit (if it exists)

$$\frac{\partial f}{\partial x_i}(p) = \lim_{t \to 0} \frac{f(p + te_i) - f(p)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(p + te_i)$$

where $e_i = (\underbrace{0, \ldots, 0}_{i-1 \text{ terms}}, i, \underbrace{0, \ldots, 0}_{n-i \text{ terms}})$.

- In $\mathbb{R}^2$, $e_1 = (1, 0)$, $e_2 = (0, 1)$.
- In $\mathbb{R}^3$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.

Partial Derivatives.

- For $n = 2$, $p = (x, y)$ and

$$\frac{\partial f}{\partial x}(x,y) = \left.\frac{d}{dt}\right|_{t=0} f(x+t, y)$$
$$\frac{\partial f}{\partial y}(x,y) = \left.\frac{d}{dt}\right|_{t=0} f(x, y+t)$$

- For $n = 3$, $p = (x, y, z)$ and

$$\frac{\partial f}{\partial x}(x,y,z) = \left.\frac{d}{dt}\right|_{t=0} f(x+t, y, z)$$
$$\frac{\partial f}{\partial y}(x,y,z) = \left.\frac{d}{dt}\right|_{t=0} f(x, y+t, z)$$
$$\frac{\partial f}{\partial z}(x,y,z) = \left.\frac{d}{dt}\right|_{t=0} f(x, y, z+t$$

Partial Derivatives.

- Do some examples of derivatives
- $f = x^2y + y^3z.$
- $f = e^{xy^2}$
- etc.

Example.

- In Economics, the partial derivatives of a utility function are called 'marginal utilities', the partial derivatives of a production function are called 'marginal products'.
- Consider, for example the Cobb-Douglas production function

$$f(K,L) = 5K^{1/3}L^{2/3}$$

  where f is the number of units produced, K is the capital and L is labor.
- The above formula means that if we use K units of capital and L units of labor, then we produce $f(K,L) = 5K^{1/3}L^{2/3}$ units of a good.
- The constants $A = 5$, $\alpha = 1/3$ and $\beta = 2/3$ are technological parameters.

Example.

- The ‘marginal products’ with respect to capital and labor are

$$\begin{aligned} \frac{\partial f}{\partial K} &= \frac{5}{3} K^{-2/3} L^{2/3} \\ \frac{\partial f}{\partial L} &= \frac{10}{3} K^{1/3} L^{-1/3} \end{aligned}$$

- The marginal product of labor,

$$\frac{\partial f}{\partial L}(K, L)$$

is interpreted in Economics as an **approximation** to the variation in the production of the good when we are using K units of capital and L units of labor and we switch to use an additional unit $L+1$ of labor and the same units K of capital as before.

Example.

- For example,
- $f(101, 100) \approx 501.661$,
- $f(100, 101) \approx 503.327$,
- $f(100, 100) \approx 500$.
- $f(101, 100) - f(100, 100) \approx 1.661, \quad \frac{\partial f}{\partial K}(100, 100) \approx 1.666$
- $f(100, 101) - f(100, 100) \approx 3.327, \quad \frac{\partial f}{\partial L}(100, 100) \approx 3.333$
- We see that the marginal product of labor and capital is positive. That is, if we use more labor and/or more capital, production increases.

Example.

- On the other hand, the marginal product of labor is decreasing in labor and increasing in capital. Suppose that we keep constant the amount of capital that we are using K. If $L' > L$ then

$$f(K, L'+1) - f(K, L') < f(K, L+1) - f(K, L)$$

- For example:
- $f(100, 101) - f(100, 100) = 3.327.$
- But, $f(100, 1001) - f(100, 1000) = 1.546.$
- We have used:
- $f(100, 100) = 500$, $f(100, 101) = 503.327$,
- $f(100, 1000) = 2320.794$, $f(100, 1001) = 2322.341$.

Example.

- That is, if we keep the capital constant, using an additional unit of labor, if we are already using a lot of labor, does not increase much the production.
- We may imagine that $f(K, L)$ is the production of a farm product in a piece of land where L is the number of the workers and the size K of the land is constant. The impact in the production when hiring and additional person is greater if few people are working in the land as compared with the case in which we already have a lot of people working in the land.

Example.

- Suppose that the amount of labor L is kept constant. If $K' > K$ then

$$f(K', L+1) - f(K', L) > f(K, L+1) - f(K, L)$$

  That is, the increase in the production when we use one additional unit of labor is larger the more capital we use. Capital and labor are complementary. In the previous example, hiring and additional worker has a larger effect on the production the larger is the size of land.
- For example:
- $f(100, 101) - f(100, 100) = 3.327$.
- But, $f(1000, 101) - f(1000, 1000) = 7.169$.
- We have used:
- $f(100, 100) = 500$, $f(100, 101) = 503.327$,
- $f(1000, 100) = 1077.217$, $f(1000, 101) = 1084.386$.

Gradient.

- The **gradient** of f at p is $\nabla f(p) = \left(\frac{\partial f}{\partial x_1}(p),\ \frac{\partial f}{\partial x_2}(p), \cdots, \frac{\partial f}{\partial x_n}(p) \right)$. Assuming all the partial derivatives exist.
- $f = x^2y + y^3z$, Compute $\nabla f(1, -1, 2)$.
- $f = e^{xy^2}$, Compute $\nabla f(-1, 1)$.

Differentiability.

- f is differentiable at p if $\lim_{v\to 0} \frac{f(p+v)-f(p)-\nabla f(p)\cdot v}{\|v\|} = 0$.
- $f(x) = (f_1(x), f_2(x), \cdots, f_m(x)) : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at p if each of the functions $f_1(x), f_2(x), \cdots, f_m(x)$ is differentiable at p.
- $n = 2$, f is differentiable at $p = (a, b)$ if $\lim_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)-\nabla f(a,b)\cdot(x-a,y-b)}{\|(x-a,y-b)\|} =$ $\lim_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)-\frac{\partial f}{\partial x}(a,b)\cdot(x-a)-\frac{\partial f}{\partial y}(a,b)\cdot(y-b)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$.

Proposition

If $f : D \to \mathbb{R}$ is differentiable at $p \in D$, then f is continuous at that point.

Example.

Consider the function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

- We will show that f is not differentiable at the point $p = (0,0)$.
- First of all, we compute $\nabla f(0,0)$. Note that

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{0}{t^3} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t \to 0} \frac{0}{t^3} = 0$$

so, $\nabla f(0,0) = (0,0)$.

Example.

- Let us use the notation $v = (x, y)$.
- Then, f is differentiable at the point $p = (0, 0)$ if and only if

$$
\begin{aligned}
0 &= \lim_{v \to 0} \frac{f(p+v) - f(p) - \nabla f(p) \cdot v}{\|v\|} \\
&= \lim_{(x,y)\to(0,0)} \frac{f((0,0)+(x,y)) - f(0,0) - \nabla f(p) \cdot (x,y)}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (0,0) \cdot (x,y)}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}}
\end{aligned}
$$

Example.

- We prove that the above limit does not exist. Consider the function

$$g(x,y) = \frac{xy^2}{(x^2+y^2)^{3/2}}$$

- Note that

$$\lim_{t \to 0} g(t,0) = \lim_{t \to 0} \frac{0}{(2t^2)^{3/2}} = 0$$

- and note that

$$\lim_{t \to 0} g(t,t) = \lim_{t \to 0} \frac{t^3}{(2t^2)^{3/2}} = \frac{1}{(2)^{3/2}} \neq 0$$

- So, the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}}$$

  does not exist and we conclude that f is not differentiable at the point $(0,0)$.

Example.

Consider now the function

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

- We will show that f is differentiable at the point $p = (0,0)$.
- First of all, we compute $\nabla f(0,0)$. Note that

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t^3} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t^3} = 0$$

- So, $\nabla f(0,0) = (0,0)$.

Example.

- Let us use the notation $v = (x, y)$.
- Then, f is differentiable at the point $p = (0, 0)$ if and only if

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0 &= \lim_{v \to 0} \frac{f(p+v) - f(p) - \nabla f(p) \cdot v}{\|v\|} \\
&= \lim_{(x,y)\to(0,0)} \frac{f((0,0)+(x,y)) - f(0,0) - \nabla f(p)\cdot(x,y)}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (0,0)\cdot(x,y)}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{xy^3}{(x^2+y^2)^{3/2}}
\end{aligned}
$$

Example.

- In the squeeze Theorem take $g(x, y) = 0$, $h(x, y) = |y|$. Since, h is continuous, we see that $\lim_{(x,y)\to(0,0)} h(x, y) = 0$.
- For $(x, y) \neq (0, 0)$ we have that

$$
\begin{aligned}
0 \leq \left| \frac{xy^3}{(x^2+y^2)^{3/2}} \right| &= \frac{|x|\, y^2\, |y|}{(x^2+y^2)^{3/2}} \\
&= \frac{\sqrt{x^2}\, y^2\, |y|}{(x^2+y^2)^{3/2}} \\
&\leq \frac{\sqrt{x^2+y^2}\, \left(x^2+y^2\right)\, |y|}{(x^2+y^2)^{3/2}} \\
&= \frac{\left(x^2+y^2\right)^{3/2}\, |y|}{(x^2+y^2)^{3/2}} \\
&= |y|
\end{aligned}
$$

Example.

- Hence,

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{(x^2+y^2)^{3/2}} = 0$$

- And the function is differentiable at the point $(0,0)$.

Differentiability.

Theorem

Suppose that there is some $r > 0$ such that the partial derivatives, $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}$ exist at every point of the open ball $B(p, r)$ and are continuous functions on that ball. Then, the function f is differentiable at p.

- f is of class C^1 in D if all the partial derivatives of f exist and are continuous functions on D. We write $f \in C^1(D)$.
- For example, $f(x, y, z) = xe^{yz} + y^2$ is differentiable at any $(x, y) \in \mathbb{R}^2$.