Session 5
Mathematics for Economics I

Continuous Functions of several variables. Extreme points

Degrees in Economics, International Studies–Economics and Law–Economics

Universidad Carlos III de Madrid

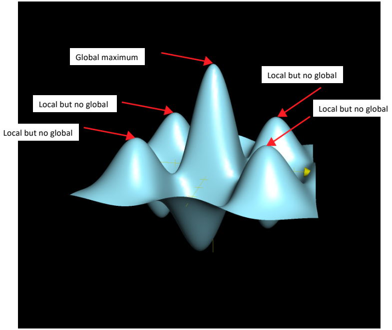
Extreme points.

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$. We say that a point $p \in D$ is a

- **global maximum** of f on D if $f(x) \leq f(p)$, for any other $x \in D$.
- **global minimum** of f on D if $f(x) \geq f(p)$, for any other $x \in D$.
- **local maximum** of f on D if there is some $\delta > 0$ such that $f(x) \leq f(p)$, for every $x \in D \cap B(p, \delta)$.
- **local minimum** of f on D if there is some $\delta > 0$ such that $f(x) \geq f(p)$, for every $x \in D \cap B(p, \delta)$.

Remark: The correct way of phrasing the previous definitions is perhaps: 'On D the function f attains a global maximum at the point p', etc. But, we will use the above shorter wording.

Extreme points.



Weierstrass' Theorem

Theorem

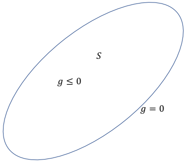
Let $D \subset \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ and let $f : D \to \mathbb{R}$ be continuous. Then, there are $x_0, x_1 \in D$ such that for any $x \in D$

$$f(x_0) \leq f(x) \leq f(x_1)$$

That is, x_0 is a global minimum of f on D and x_1 is a global maximum of f on D.

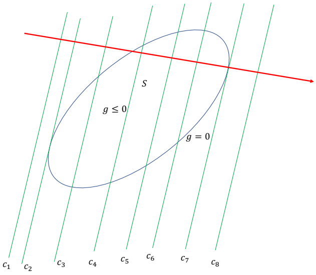
Level Curves and extreme points

- Using the level curves, it is possible to find the extreme points of a function f in a set S.
- We explain this in two dimensions.
- Let f be a function of two variables.
- Let $S \subset \mathbb{R}^2$.
- We want to find the maximum and minimum values of f on S.
- Suppose $S = \{(x, y) \in \mathbb{R}^2 : g(x, y) \leq 0\}$ and $\partial S = \{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\}$.



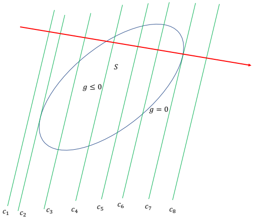
Level Curves and extreme points

- In green we have the level curves of the function f. For simplicity, they are depicted as straight lines.
- The red arrow indicates the direction of growth of the level curves. That is $c_1 < c_2 < \cdots < c_8$.



Level Curves and extreme points

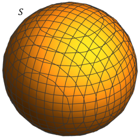
- Neither the maximum nor the minimum values can be attained a level curve that intersects ∂S at two points.
- For example, moving from a point in the level of curve c_5 to a point in the level of curve c_6 increases the value of f. And moving from a point in the level of curve c_5 to a point in the level of curve c_4 decreases the value of f.



- The maximum value is c_7. It is attained at a point where the level curve is tangent to the curve $g = 0$.
- The minimum value is c_2. It is attained at a point where the level curve is tangent to the curve $g = 0$.

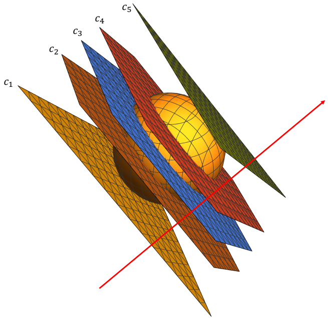
Level Curves and extreme points

- The same idea works in any dimension. For example, Let f be a function of thee variables and $S \subset \mathbb{R}^3$.
- We want to find the maximum and minimum values of f on S.
- Suppose $S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) \leq 0\}$ and $\partial S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$. Note that now ∂S is a surface in $\mathbb{R}^3$.



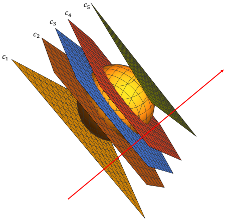
Level Curves and extreme points

- For simplicity, he level curves of the function f are depicted as planes in $\mathbb{R}^3$.
- The red arrow indicates the direction of growth of the level curves. That is, $c_1 < c_2 < \cdots < c_5$.



Level Curves and extreme points

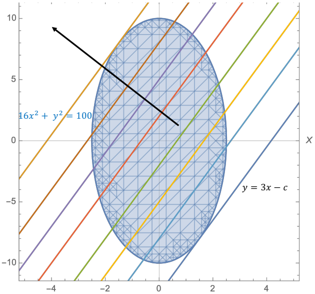
- Neither the maximum nor the minimum values can be attained a level curve that intersects ∂S at a curve.
- For example, moving from a point in the level of curve c_3 to a point in the level of curve c_4 increases the value of f. And moving from a point in the level of curve c_3 to a point in the level of curve c_2 decreases the value of f.



- The maximum value is c_5. It is attained at a point where the level curve is tangent to the surface $g = 0$.
- The minimum value is c_1. It is attained at a point where the level curve is tangent to the surface $g = 0$.

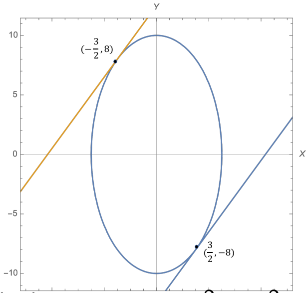
Example 1

- Consider the set $A = \{(x, y) \in \mathbb{R}^2 : 16x^2 + y^2 \leq 100\}$ and the function $f(x, y) = y - 3x$.
- The function is continuous and the set A is closed. It is also bounded and hence the set A is compact.
- Therefore, the function f attains a maximum and a minimum on A.
- The level curves of f are lines of the form $y = 3x + C$. Graphically,



Example 1.

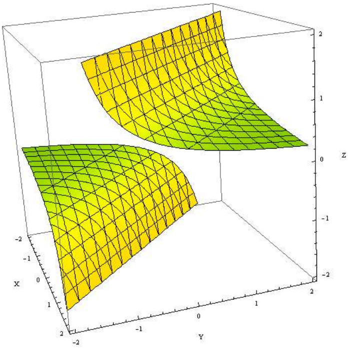
- The maximum and the minimum value are attained at the point (x_0, y_0) where the line $y = 3x + C$ is tangent to the graph of $16x^2 + y^2 = 100$.



- We have $32x_0 + 2y_0 y'(x_0) = 0$, $y'(x_0) = 3$ and $16x_0^2 + y_0^2 = 100$.
- We obtain the points $\left(-\frac{3}{2}, 8\right)$ (corresponds to a maximum) and $\left(\frac{3}{2}, -8\right)$ (corresponds to a minimum).
- The maximum value is $\frac{25}{2}$. The minimum value is $-\frac{25}{2}$.

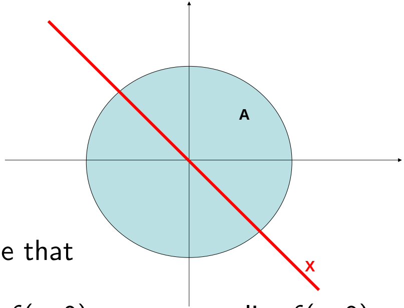
Example 2

Consider the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\}$ and the function $f(x, y) = \frac{1}{x+y}$. The graph of f is



Example 2

The function f is continuous except in the set $X = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$. This set intersects A,



Taking $y = 0$, we see that

$$\lim_{\substack{x \to 0 \\ x > 0}} f(x, 0) = +\infty \qquad \lim_{\substack{x \to 0 \\ x < 0}} f(x, 0) = -\infty$$

and we conclude that f attains neither a maximum nor a minimum on the set A.

Example 3.

Consider the set $B_1 = \{(x, y) \in \mathbb{R}^2 : xy \geq 1, \quad x, y > 0\}$ and function $f(x, y) = \frac{1}{x+y}$.

- The function $f(x, y) = \frac{1}{x+y}$ is continuous on B_1, because if $y = -x$, then $xy = -x^2 \leq 0$.
- The set B_1 is closed but not bounded. Hence, it is not compact.
- We may not apply Weierstrass' theorem.
- On the one hand, we see that $f(x, y) > 0$ in the set B_1.
- In addition, the points (n, n) for $n = 1, 2, \ldots$ belong to the set B_1 and

$$\lim_{n \to +\infty} f(n, n) = 0$$

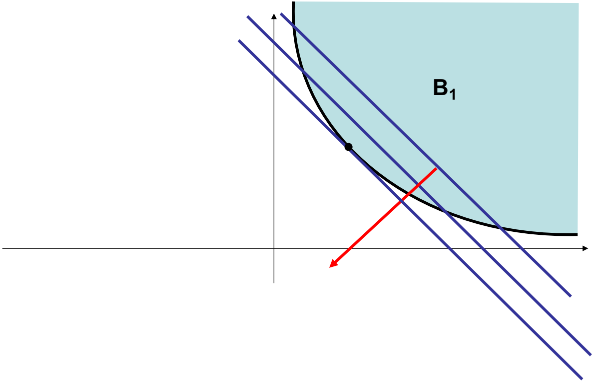
- Hence, given a point $p \in B_1$, we may find a natural number n large enough such that

$$f(p) > f(n, n) > 0$$

And we conclude that f does not attain a minimum in the set B_1.

Example 3

- On the other hand, the level curves function are the straight lines $x + y = \frac{1}{c}$.
- Graphically, we see that f attains the maximum value at the point $p = (a, b)$ where the line $x + y = \frac{1}{c}$ is tangent to the curve $xy = 1$.



Example 3

- The slope of the line $x + y = \frac{1}{c}$ is $m = -1$. Why?
- The point $p = (a, b)$ is on the curve $xy = 1$. Hence, $ab = 1$
- To compute the tangent line at $p = (a, b)$ we differentiate implicitly $xy = 1$ to obtain $y + xy' = 0$.
- We plug in $y' = -1$, $x = a$, $y = b$ and get $b - a = 0$.
- We have $a = b$, $ab = 1$ and $a, b > 0$. Thus, $a = b = 1$.
- The maximum value is attained at the point $(1, 1)$.