Session 4
Mathematics for Economics I

Continuous Functions of several variables

Grados en Economía, Estudios Internacionales–Economía y Derecho–Economía

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Continuous Functions.

- A function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at a point $p \in D$ if $\lim_{x \to p} f(x) = f(p)$.
- We say that f is continuous on D if its continuous at every point $p \in D$.
- The function $f(x) = (f_1(x), \ldots, f_m(x))$ is continuous at $p \in D$ if and only if for each $i = 1, \ldots, m$, the function f_i are continuous at p.
- We will concentrate on functions $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Examples of Continuous Functions.

The following functions are continuous,

- Polynomials
- Trigonometric and exponential functions.
- Logarithms, in the domain where is defined.
- Powers of funcions, in the domain where they are defined.
- Algebraic combinations of the above functions.

Examples of Continuous Functions.

Theorem

Let $D \subset \mathbb{R}^n$ and let $f, g : D \to \mathbb{R}$ be continuous at a point p in D. Then,

1. *$f + g$ is continuous at p.*
2. *fg is continuous at p.*
3. *if $f(p) \neq 0$, then there is some open set $U \subset \mathbb{R}^n$ such that $f(x) \neq 0$ for every $x \in U \cap D$ and*

$$\frac{g}{f} : U \cap D \to \mathbb{R}$$

is continuous at p.

Theorem

Let $f : D \subset \mathbb{R}^n \to E$ (where $E \subset \mathbb{R}^m$) be continuous at $p \in D$ and let $g : E \to \mathbb{R}^k$ be continuous at $f(p)$. Then, $g \circ f : D \to \mathbb{R}^k$ is continuous at p.

Continuity of functions and open/closed sets

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$. Then, the following are equivalent.

1. *f is continuous on $\mathbb{R}^n$.*
2. *For each open subset U of $\mathbb{R}$, the set $f^{-1}(U) = \{x \in \mathbb{R}^n : f(x) \in U\}$ is open.*
3. *For each closed subset $V \subset \mathbb{R}$, the set $\{x \in \mathbb{R}^n : f(x) \in V\}$ is closed.*
4. *For each $a, b \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n : a \leq f(x) \leq b\}$ is closed.*
5. *For each $a, b \in \mathbb{R}$, the set $f^{-1}(a, b) = \{x \in \mathbb{R}^n : a < f(x) < b\}$ is open.*

Continuity of functions and open/closed sets

Corollary

Suppose that the functions $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ are continuous. Let $-\infty \leq a_i \leq b_i \leq +\infty$, $i = 1, \ldots, k$. Then,

1. The set $\{x \in \mathbb{R}^n : a_i < f_i(x) < b_i, \quad i = 1, \ldots, k\}$ is open.
2. The set $\{x \in \mathbb{R}^n : a_i \leq f_i(x) \leq b_i, \quad i = 1, \ldots, k\}$ is closed.

Example 1.

Consider the function

$$\begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We prove that it is continuous in $\mathbb{R}^2$.

- The function f is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$ because there it is a quotient of polynomials and the denominator does not vanish in $\mathbb{R}^2 \setminus \{(0,0)\}$.
- To prove the continuity at $(0,0)$ we prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.
- In the squeeze Theorem, we let $g(x,y) = 0$, $h(x,y) = |x|$.
- The functions g and h are continuous so $\lim_{(x,y)\to(0,0)} g(x,y) = 0$ and $\lim_{(x,y)\to(0,0)} h(x,y) = h(0,0) = 0$.

Example 1.

- Finally, we have,

$$|f(x,y)| = |x|\frac{y^2}{x^2+y^2} \leq |x|$$

- By the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$.
- And, since, $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$, we apply again the Squeeze Theorem to conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

- Hence, f is also continuous at $(0,0)$.

Example 2.

Consider the function

$$f(x,y) = \begin{cases} \frac{x^2\sqrt{|y|}}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We prove that it is continuous in $\mathbb{R}^2$.

- The function f is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$ because in $\mathbb{R}^2 \setminus \{(0,0)\}$ f is a quotient of continuous functions and the denominator does not vanish in $\mathbb{R}^2 \setminus \{(0,0)\}$.
- To prove the continuity at $(0,0)$ we prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.
- In the squeeze Theorem, we let $g(x,y) = 0$, $h(x,y) = \sqrt{|y|}$.
- The functions g and h are continuous so $\lim_{(x,y)\to(0,0)} g(x,y) = 0$ and $\lim_{(x,y)\to(0,0)} h(x,y) = h(0,0) = 0$.

Example 2.

- Finally, we have,

$$|f(x,y)| = \frac{x^2\sqrt{|y|}}{x^2+y^2} = \frac{x^2}{x^2+y^2}\sqrt{|y|} \leq \sqrt{|y|}$$

- By the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$.
- And, since, $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$, we apply again the Squeeze Theorem to conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

- Hence, f is also continuous at $(0,0)$.

Example 3.

Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

We prove that it is continuous in $\mathbb{R}^2$.

- The function f is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$ because in $\mathbb{R}^2 \setminus \{(0,0)\}$ f is a quotient of continuous functions and the denominator does not vanish in $\mathbb{R}^2 \setminus \{(0,0)\}$.
- To prove the continuity at $(0,0)$ we prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.
- In the squeeze Theorem, we let $g(x,y) = 0$, $h(x,y) = \sqrt{x^2+y^2}$.
- The functions g and h are continuous so $\lim_{(x,y)\to(0,0)} g(x,y) = 0$ and $\lim_{(x,y)\to(0,0)} h(x,y) = h(0,0) = 0$.

Example 3.

- Finally, we have,
  $|f(x,y)| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}.$
- By the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} |f(x,y)| = 0.$
- And, since, $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$, we apply again the Squeeze Theorem to conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

- Hence, f is also continuous at $(0,0)$.

Example 4.

Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} e^{xy} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Determine the subset of $\mathbb{R}^2$ where the function f is continuous.

- The function f is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$ because in $\mathbb{R}^2 \setminus \{(0,0)\}$ f is a quotient of continuous functions and the denominator does not vanish in $\mathbb{R}^2 \setminus \{(0,0)\}$.
- We prove that f is not continuous at $(0,0)$.
- Consider the curve $\alpha(t) = (t,t)$ we obtain $\lim_{t\to 0} f(\alpha(t)) = \lim_{t\to 0} \frac{t^2}{2t^2} e^{t^2} = \frac{1}{2}$.
- If we use now the curve $\sigma(t) = (t,t^2)$ we see that $\lim_{t\to 0} f(\sigma(t)) = \lim_{t\to 0} \frac{t^3}{t^2+t^4} e^{t^3} = \lim_{t\to 0} \frac{t}{1+t} e^{t^3} = 0$.
- Hence the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.
- The function f is not continuous at the point $(0,0)$.