# Session 3
# Mathematics for Economics I

Functions of several variables

Degrees in Economics, International Studies and Economics and Law and Economics

Universidad Carlos III de Madrid

# Limits of functions.

**Definition**

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and let $L \in \mathbb{R}$, $p \in \mathbb{R}^n$. We say that

$$\lim_{x \to p} f(x) = L$$

if given $\varepsilon > 0$ there is some $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $0 < \|x - p\| < \delta$.

- This is the natural generalization of the concept of limit for one-variable functions to functions of several variables, once we remark that the distance $|\cdot|$ in $\mathbb{R}$ is replaced by the distance $\|\cdot\|$ in $\mathbb{R}^n$).
- The interpretation is the same, i.e., $|x - y|$ is the distance from $x$ to $y$ in $\mathbb{R}$ and $\|x - y\|$ is the distance from $x$ to $y$ in $\mathbb{R}^n$.

# Remarks about limits.

- The limit of a function at a point does not always exist.
- If the limit exists, it is unique.
- The calculus of limits with several variables is more complicated than the calculus of limits with one variable.

# Showing that the limit exists. The easy cases

## Proposition (4)

*Suppose $f$ is a function of the following type:*

1. *A polynomial, or an absolute value, or an exponential function, or a logarithm, or a trigonometric function or an irrational function.*
2. *A composition or an algebraic combination of the above functions.*

*Let $p$ be in the domain of $f$. Then*

$$\lim_{x \to p} f(x) = f(p)$$

# Example

Let $f(x,y) = \frac{xy^2 + e^{2x-y} + \ln(x^2+y)}{x^2 y}$. Then

$$\begin{aligned}
\lim_{(x,y)\to(1,2)} f(x) &= \left. \frac{xy^2 + e^{2x-y} + \ln(x^2+y)}{x^2 y} \right|_{x=1,y=2} \\
&= \frac{4 + e^0 + \ln(1+2)}{2} \\
&= \frac{5 + \ln 3}{2}
\end{aligned}$$

# Two useful tools to show that the limit exists.

The following results will be in proving that a limit exists.

## Proposition

**(Squeeze Theorem)** *Let* $f, g, h : \mathbb{R}^n \to \mathbb{R}$ *and suppose*

1. $g(x) \leq f(x) \leq h(x)$ *for every* $x$ *in some open disc centered at* $p$.
2. $\lim_{x\to p} g(x) = \lim_{x\to p} h(x) = L$.

*Then,*

$$\lim_{x\to p} f(x) = L$$

## Proposition

*The following inequalities hold.*

1. $|xy| \leq \frac{1}{2}\left(x^2 + y^2\right) \leq x^2 + y^2$.
2. $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$.
3. $|y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2}$.

# Example

Consider the function $f(x, y) = \begin{cases} (x^2 + y^2)\cos(\frac{1}{x^2+y^2}) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$.

- Note that for $(x, y) \neq (0, 0)$: $\left|\cos(\frac{1}{x^2+y^2}\right| \leq 1$.
- Thus, $0 \leq |f(x, y)| \leq x^2 + y^2$.
- In the previous proposition we take $g(x, y) = 0$, $h(x, y) = x^2 + y^2$.
- We have $\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{(x,y)\to(0,0)} h(x, y) = 0$.
- We conclude $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

# Example

Consider the function $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ .

- Note that $|xy| \leq \sqrt{x^2+y^2}\sqrt{x^2+y^2} = x^2+y^2$. And, for $(x,y) \neq (0,0)$:
- $0 \leq \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$.
- Thus, $0 \leq |f(x,y)| \leq \sqrt{x^2+y^2}$.
- In the previous proposition we take $g(x,y) = 0$, $h(x,y) = \sqrt{x^2+y^2}$.
- We have $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} h(x,y) = 0$.
- We conclude $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

# Example

Let us consider the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

- Consider the functions

$$g(x,y) = 0, \quad h(x,y) = \sqrt{x^2+y^2}$$

- By Proposition 4, we have $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} h(x,y) = 0.$

# Example

- On the other hand,

$$|f(x,y)| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

- So, $g(x,y) \leq |f(x,y)| \leq h(x,y)$
- By the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$.
- And, since, $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$, we apply again the Squeeze Theorem to conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

# Some tools to show that the limit does not exist: Iterated limits

- Suppose that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and that the following one-dimensional limits

$$\lim_{x\to a} f(x,y)$$
$$\lim_{y\to b} f(x,y)$$

exist for $(x,y)$ in a ball around $(a,b)$.

- Define the functions

$$g_1(y) = \lim_{x\to a} f(x,y) \quad g_2(x) = \lim_{y\to b} f(x,y)$$

- Then,

$$\lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right) = \lim_{x\to a} g_2(x) = L$$
$$\lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) = \lim_{y\to b} g_1(y) = L$$

# Iterated limits

- If we know beforehand that it exists, we may use the iterated limits to compute its value.
- Or, if for some function $f(x, y)$ we can prove that

  $$\lim_{x \to a} \lim_{y \to b} f(x, y) \neq \lim_{y \to b} \lim_{x \to a} f(x, y)$$

  then $\lim_{(x,y) \to (a,b)} f(x, y)$ does not exist.
- However, **iterated limits may not be used to prove that a limit exists.**

# Iterated limits. Example

Consider the function,

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

- Note that

$$\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

- But,

$$\lim_{y\to 0} \lim_{x\to 0} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

- Hence, the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

does not exist.

# Other tools to show that a limit does not exist. Limits through curves.

## Proposition

*Let* $p \in D \subset \mathbb{R}^n$ *and* $f : D \subset \mathbb{R}^n \to \mathbb{R}$*. Consider a curve* $\sigma : [-\varepsilon, \varepsilon] \to D$ *such that* $\sigma(0) = p$ $\sigma(t) \neq p$ *whenever* $t \neq 0$ *and* $\lim_{t \to 0} \sigma(t) = p$*. Suppose,* $\lim_{x \to p} f(x) = L$*. Then,*

$$\lim_{t \to 0} f(\sigma(t)) = L$$

## Limits through curves. Example

Consider the function,

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

- Note that the iterated limits

$$\lim_{x\to 0} \lim_{y\to 0} f(x,y) = \lim_{x\to 0} \frac{0}{x^2} = 0$$

$$\lim_{y\to 0} \lim_{x\to 0} f(x,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$$

  coincide.

- But, if we consider the curve, $\sigma(t) = (t,t)$, the limit

$$\lim_{t\to 0} f(\sigma(t)) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0} \frac{t^2}{2t^2} = \frac{1}{2}$$

  does not coincide with the value of the iterated limits.

- Hence, the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

# Limits through curves. Example

Consider the function,

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

- Note that the iterated limits

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{x^4} = 0$$

and

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$$

coincide.
- But, if we consider the curve, $\sigma(t) = (t, t^2)$, the limit

$$\lim_{t\to 0} f(t,t^2) = \lim_{x\to 0} f(t,t^2) = \lim_{t\to 0} \frac{t^4}{t^4+t^4} = \frac{1}{2}$$

does not coincide with the value of the iterated limits.
- Hence, the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exist.

# Algebra of limits

Consider two functions $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$ and suppose

$$\lim_{x \to p} f(x) = L_1, \quad \lim_{x \to p} g(x) = L_2$$

Then,

1. $\lim_{x \to p} (f(x) + g(x)) = L_1 + L_2$.
2. $\lim_{x \to p} (f(x) - g(x)) = L_1 - L_2$.
3. $\lim_{x \to p} f(x)g(x) = L_1 L_2$.
4. If $a \in \mathbb{R}$ then $\lim_{x \to p} af(x) = aL_1$.
5. If, in addition, $L_2 \neq 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$$