Session 2
Mathematics for Economics I

Functions of several variables

Degrees in Economics, International Studies and Economics and Law and Economics

Universidad Carlos III de Madrid

Examples of functions $f : \mathbb{R}^n \to \mathbb{R}$.

- $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x + y - 1$.
- $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2 + \sqrt{1 + z^2}$.
- $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = z \exp x^2 + y^2$.
- $f : \mathbb{R}^4 \to \mathbb{R}$ defined by $f(x, y, z, t) = \sin x + y + z \exp t$.

Functions $f : \mathbb{R}^n \to \mathbb{R}^m$

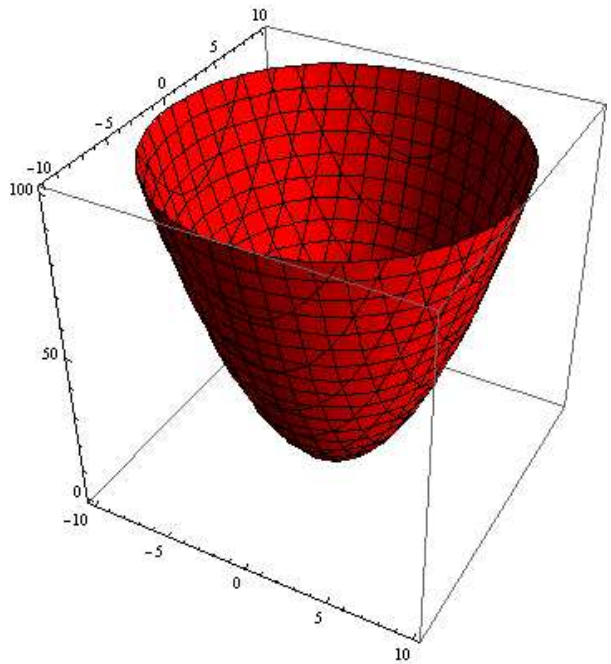
- For example, $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) = (x \exp y + \sin z, x^2 + y^2 - z^2)$.
- We may write $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$ with $f_1(x, y, z) = x \exp y + \sin z$, $f_2(x, y, z) = x^2 + y^2 - z^2$
- So, we may just focus on functions $f : \mathbb{R}^n \to \mathbb{R}$.

Implicit Domain

- When we write, for example, $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$ it is understood that $x \neq 1$.
- The expression of f defines implicitly the domain of the function.
- For the above function we need that $x + y + 1 \geq 0$ and $x \neq 1$.
- So, we assume implicitely that the domain of $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$ is the set $D = \{(x,y) \in \mathbb{R}^2 : x + y \geq -1, x \neq 1\}$.
- Usually, we will write $f : D \subset \mathbb{R}^n \to \mathbb{R}$ to make explicit the domain of f.

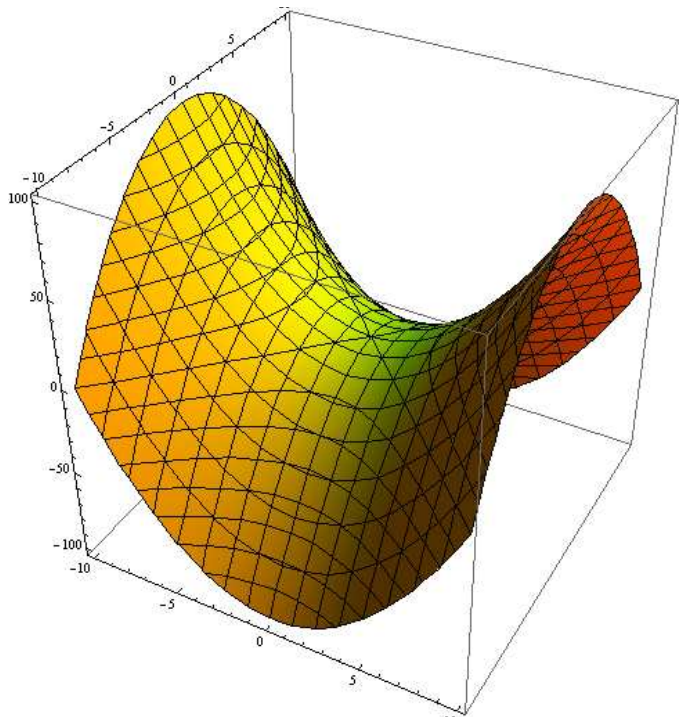
Graph of a function of several variables.

- The **graph** of $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is $G(f) = \{(x, y) \in \mathbb{R}^{n+1} : y = f(x), x \in D\}$.
- The graph can be drawn only for $n = 1, 2$.
- The graph of $f(x, y) = x^2 + y^2$ is



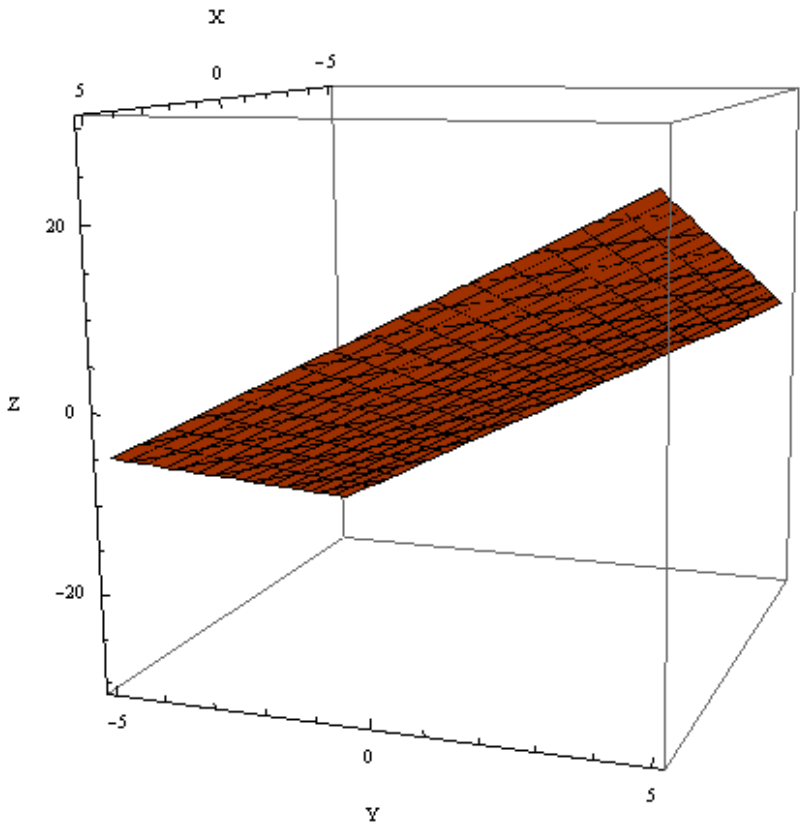
Example

The graph of $f(x, y) = -x^2 + y^2$ is



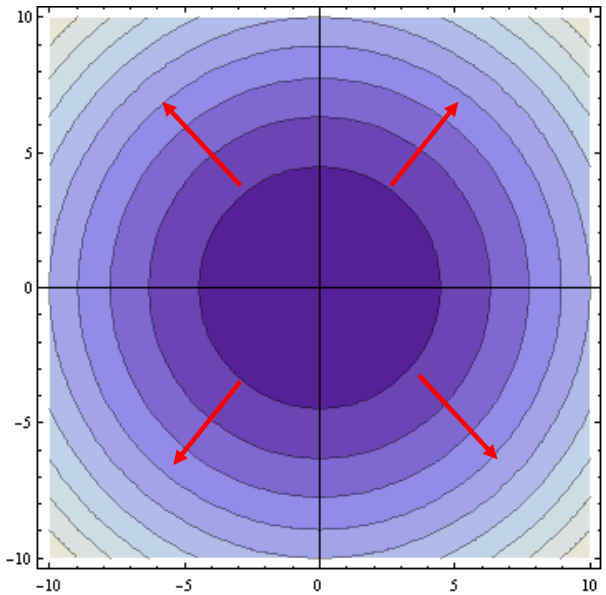
Example

The graph of $f(x, y) = 2x + 3y$ is



Level curves and level surfaces

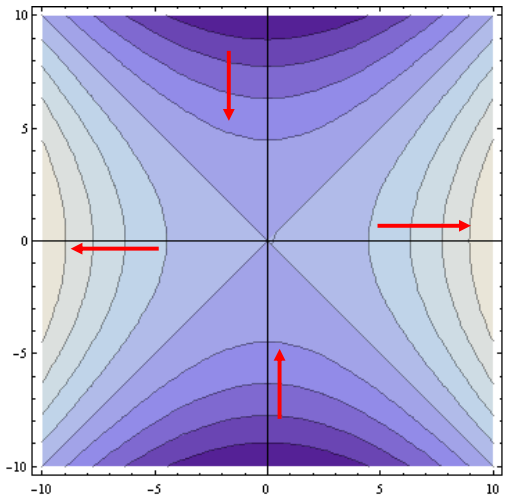
- Given $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and $k \in \mathbb{R}$ we define the **level surface** of f as the set $C_k = \{x \in D : f(x) = k\}$.
- If $n = 2$, the level surface is called a **level curve.**
- Example. The level curves of $f(x, y) = x^2 + y^2$ are



- The arrows point in the direction in which the function f grows.

Example

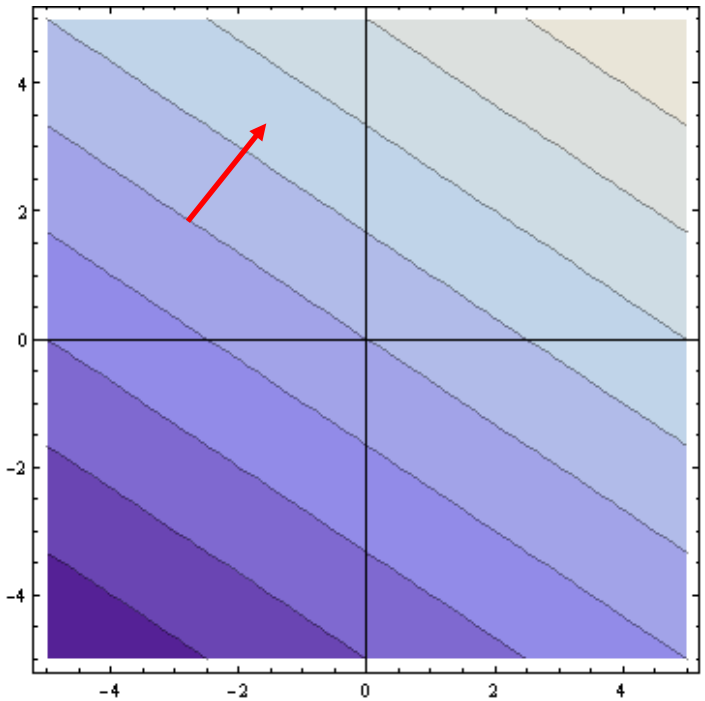
- The level curves of $f(x, y) = x^2 - y^2$ are



- The arrows point in the direction in which the function f grows.

Example

- The level curves of $f(x, y) = 2x + 3y$ are



- The arrows point in the direction in which the function f grows.

Example

- For $c \geq 0$, the level curves of $f(x, y) = (2x + 3y)^2$ are given by

$$(2x + 3y)^2 = c$$

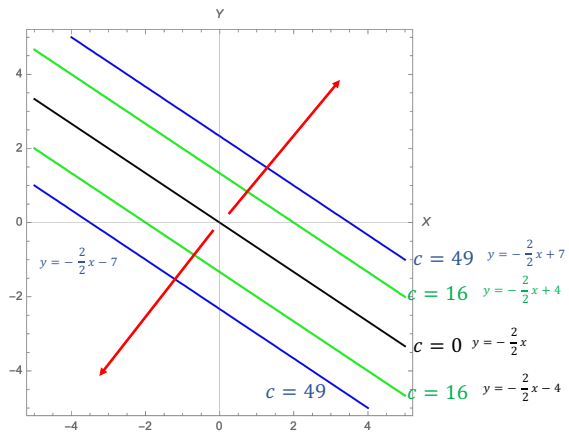
That is,

$$2x + 3y = \pm\sqrt{c}$$

- We obtain two lines

$$y = -\frac{2}{3}x + \sqrt{c} \quad \text{and} \quad y = -\frac{2}{3}x - \sqrt{c}$$

graphically,



Example

- And the graph is the following