Session 14
Mathematics for Economics II

Concave/convex functions. Examples and applications.

Degrees in Business Administration, Finance and Accounting, Management and Technology, International Studies and Business Administration and Law and Business Administration

Universidad Carlos III de Madrid

Example.

- Let $f(x, y, z) = 2abyz + ax^2 + 2axy + 2ay^2 + cz^2 + 3x + y + 15z - 73$, with $a, b, c \in \mathbb{R}$, $abc \neq 0$. For which values of $a, b, c \in \mathbb{R}$, $abc \neq 0$ is the function convex/concave?
- We have $\nabla f(x, y, z) =$ $(2ax + 2ay + 3, 2abz + 2ax + 4ay + 1, 2aby + 2cz + 15)$.

$$\mathrm{H}(f)(x, y, z) = \begin{pmatrix} 2a & 2a & 0 \\ 2a & 4a & 2ab \\ 0 & 2ab & 2c \end{pmatrix}$$

- We obtain $D_1 = 2a$, $D_2 = 4a^2 > 0$, $D_3 = |A| = 8a^2c - 8a^3b^2 = 8a^2\left(c - ab^2\right)$.
- We see that for $a > 0$ and $c > ab^2$ we have that $D_1, D_2, D_3 > 0$ and the function is convex.
- For $a < 0$ and $c < ab^2$ we have that $D_1 < 0, D_2 > 0, D_3 < 0$ and the function is concave.

Example.

- Take $a = -1,\ b = 2,\ c = -5$.
- We obtain the function
  $f(x, y, z) = -x^2 - 2xy + 3x - 2y^2 - 4yz + y - 5z^2 + 15z - 73.$
- Since $c < ab^2$, the function f is concave.
- The set
  $D = \{(x, y, z) \in \mathbb{R}^3 : -x^2 - 2xy + 3x - 2y^2 - 4yz + y - 5z^2 + 15z \geq 0\}$
  is convex.
- Since $f(x, y, z) \leq f(1, 0, 0) + \nabla f(1, 0, 0) \cdot (x - 1, y, z)$,
  $f(1, 0, 0) = -71$,
  $\nabla f(x, y, z) = (3 - 2x - 2y, 1 - 2x - 4y - 4z, 15 - 4y - 10z)$ and
  $\nabla f(1, 0, 0) = (1, -1, 15)$, we have the inequality,
  $-x^2 - 2xy + 3x - 2y^2 - 4yz + y - 5z^2 + 15z - 73 \leq$
  $-71 - 1 + x - y + 15z = -72 + x - y + 15z$, for every $x, y, z \in \mathbb{R}$.

Example. Final exam January 2021.

- Consider the function $f(x, y) = cxy + x^2y + 2x + y^2 - 15y + 1$ defined in $\mathbb{R}^2$, with $c \in \mathbb{R}$.
- What is the largest open subset D of $\mathbb{R}^2$ in which the function f is strictly convex? For what values of c is the set D convex?
- We have $\nabla f(x, y) = \big(cy + 2xy + 2, cx + x^2 + 2y - 15\big)$
  $\mathrm{H}(f)(x, y, z) = \begin{pmatrix} 2y & c + 2x \\ c + 2x & 2 \end{pmatrix}$
- $D_1 = 2y$, $D_2 = 4y - (c + 2x)^2$
- We see that $D_1 > 0$ if and only if $y > 0$ and $D_2 > 0$ if and only if $y > \frac{1}{4}(c + 2x)^2$.
- Hence,

$$D = \{(x, y) \in \mathbb{R}^2 : y > \frac{1}{4}(c + 2x)^2\}$$

Example. Final exam January 2021, continued.

- Consider the function

$$g(x) = \frac{1}{4}(c + 2x)^2$$

- Since, $g''(x) = 2 > 0$, the function g is convex.
- Therefore the set $D = \{(x, y) \in \mathbb{R}^2 : y > \frac{1}{4}(c + 2x)^2\}$ is convex for any value of c.

Examples.

- Final exam, January 2024: Prove that the set $A = \{(x, y) \in \mathbb{R}^2 : y^2 + x \leq 0 \leq x + 5\}$ is convex.
- Final exam, January 2023: Prove that the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 - 4 \leq y \leq 4 - x^2\}$ is convex.
- Final exam, January 2022: Prove that the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25, x + y \geq 5$ is convex.
- Final exam, January 2021: Prove that the set $A = \{(x, y) \in \mathbb{R}^2 : y^2 \leq x - 1, x \leq 5\}$ is convex.

Concavity, convexity and preferences.

- Lat us consider again a consumer whose preferences (over two consumption goods) are represented by the utility function $u(x, y)$.
- The indifference curves of the consumer are the sets

$$\{(x, y) \in \mathbb{R}^2 : x, y > 0, \quad u(x, y) = C\}$$

  with $C \in \mathbb{R}$.
- Suppose that the function $u(x, y)$ is differentiable and that

$$\frac{\partial u}{\partial x} > 0 \qquad \frac{\partial u}{\partial y} > 0$$

- Applying the implicit function Theorem, we see that the equation

$$u(x, y) = C$$

  defines y as a function of x.

Concavity, convexity and preferences.

- The set
$$\{(x,y) \in \mathbb{R}^2 : x,y > 0, \quad u(x,y) = C\}$$
may be represented as the graph of the function $y(x)$.
- Differentiating implicitly the equation $u(x,y) = C$ we may compute the derivative y'
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} y'(x) = 0$$
- Applying again the implicit function Theorem we obtain an equation for the second derivative y''

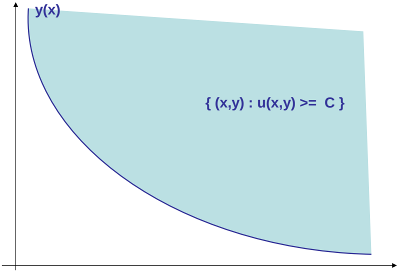
$$\frac{\partial^2 u}{\partial x \partial x} + 2\frac{\partial^2 u}{\partial x \partial y} y'(x) + \frac{\partial^2 u}{\partial y \partial y} \left(y'(x)\right)^2 + \frac{\partial u}{\partial x} y''(x) = 0 \qquad (0.1)$$

Concavity, convexity and preferences.

- One of l standard assumptions in Economic Theory is that the set which consist of all the consumption bundles which are preferred to a given consumption bundle is a convex set. In terms of the utility function this means that the set

$$\{(x, y) \in \mathbb{R}^2 : x, y > 0, \quad u(x, y) > C\}$$

is convex.



Concavity, convexity and preferences.

- Suppose that the function $u(x, y)$ is concave and of class C^2.
- Then, the set $\{(x, y) \in \mathbb{R}^2 : x, y > 0, \quad u(x, y) > C\}$ is convex.
- In addition, for every $h, k \in \mathbb{R}$ we have that

$$\frac{\partial^2 u}{\partial x \partial x} h^2 + 2 \frac{\partial^2 u}{\partial x \partial y} hk + \frac{\partial^2 u}{\partial y \partial y} k^2 \leq 0$$

- If in this equation we plug in $h = 1$, $k = y'(x)$ we obtain that

$$\frac{\partial^2 u}{\partial x \partial x} + 2 \frac{\partial^2 u}{\partial x \partial y} y'(x) + \frac{\partial^2 u}{\partial y \partial y} \left(y'(x)\right)^2 \leq 0$$

- and solving for y'' in the equation 0.1 we obtain

$$y''(x) = -\frac{\frac{\partial^2 u}{\partial x \partial x} + 2 \frac{\partial^2 u}{\partial x \partial y} y'(x) + \frac{\partial^2 u}{\partial y \partial y} \left(y'(x)\right)^2}{\partial u / \partial x} \geq 0$$

Concavity, convexity and preferences.

- that is, the function $y(x)$ is convex, so that $y'(x)$ is increasing.
- Since $\mathrm{MRS}(x, y(x)) = -y'(x)$, we see that if the preferences of the consumer are convex his marginal rate of substitution is decreasing.