Session 13
Mathematics for Economics II

Concave/convex functions. Definition and characterizations.

Degrees in Business Administration, Finance and Accounting, Management and Technology, International Studies and Business Administration and Law and Business Administration

Universidad Carlos III de Madrid

Concavity and convexity.

We assume that: $D \subset \mathbb{R}^n$ is a convex, open set.

- f is **concave** on D if for every $\lambda \in [0,1]$ and $x, y \in D$ we have that

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$$

- **convex** on D if for every $\lambda \in [0,1]$ and $x, y \in D$ we have that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

- f is convex on D if and only if $-f$ is concave on D.

Concavity and convexity.

- f is **strictly concave** on D if for every $\lambda \in (0,1)$ and $x, y \in D$, $x \neq y$ we have that

$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$$

- f is **strictly convex** on D if for every $\lambda \in (0,1)$ and $x, y \in D$, $x \neq y$ we have that

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$$

- f is strictly convex on D if and only if $-f$ is strictly concave on D.

Linear functions are concave and convex.

- Linear functions are concave and convex, but neither strictly concave nor strictly convex.
- For example, $f(x, y) = 2x - y$, $f(x, y, z) = z - 3x + 5y$ are concave and convex.

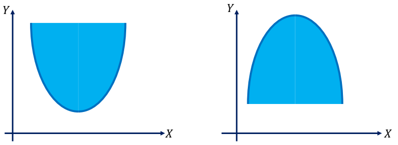
Concavity and convexity.

Proposition

Let D be a convex, open subset of $\mathbb{R}^n$. Then,

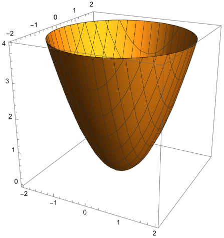
1. f is concave $\Leftrightarrow$ the set $\{(x, y) : x \in D, y \leq f(x)\}$ is convex.
2. f is convex $\Leftrightarrow$ the set $\{(x, y) : x \in D, y \geq f(x)\}$ is convex.
3. f is strictly convex $\Leftrightarrow$ the set $\{(x, y) : x \in D, y \geq f(x)\}$ is convex and the graph of f contains no segments.
4. f is strictly concave $\Leftrightarrow$ the set $\{(x, y) : x \in D, y \leq f(x)\}$ is convex and the graph of f no contains segments.

Graphically,

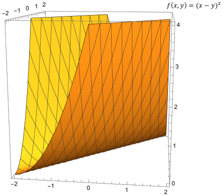


Examples.

- $f(x,y) = x^2 + y^2$ is strictly convex.



- $f(x,y) = (x-y)^2$ is convex, but not strictly convex.



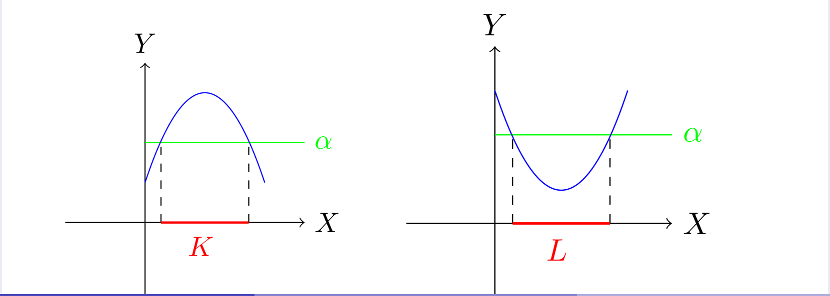
Concavity and convexity.

Proposition

Let D be a convex, open subset of $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then,

1. If f is concave, then the upper contour set $K = \{x \in D : f(x) \geq \alpha\}$ is convex for every $\alpha \in \mathbb{R}$
2. If f is convex, then the lower contour set $L = \{x \in D : f(x) \leq \alpha\}$ is convex for every $\alpha \in \mathbb{R}$

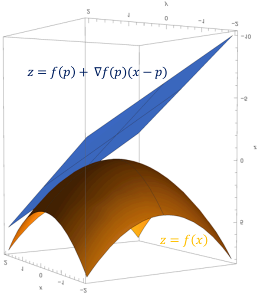
Graphically,

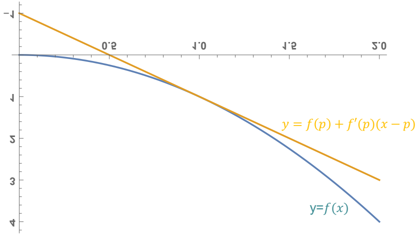


First order conditions for concavity.

- Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is concave and differentiable on a convex set D.
- The plane tangent to the graph of f at $p \in D$ is above the graph of f.
- This means

$$f(x) \leq f(p) + \nabla f(p) \cdot (x - p), \quad \text{for every } x \in D$$





First order conditions for concavity.

Proposition

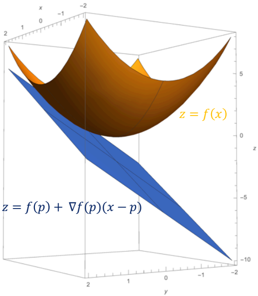
Suppose $f \in C^1(D)$. Then,

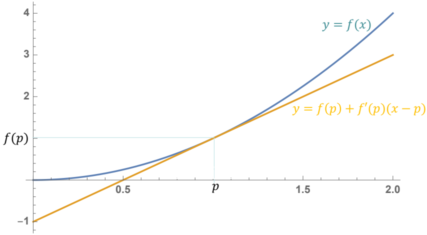
1. f is concave on D if and only if for all $u, v \in D$ we have that
$$f(u) \leq f(v) + \nabla f(v) \cdot (u - v)$$
2. f is strictly concave on D if and only if for all $u, v \in D$, $u \neq v$, we have that
$$f(u) < f(v) + \nabla f(v) \cdot (u - v)$$

First order conditions for convexity.

- Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable on a convex set D.
- The plane tangent to the graph of f at $p \in D$ is below the graph of f.
- This means

$$f(x) \geq f(p) + \nabla f(p) \cdot (x - p), \quad \text{for every } x \in D$$





First order conditions for convexity.

Proposition

Suppose $f \in C^1(D)$. Then,

1. f is convex on D if and only if for all $u, v \in D$ we have that
$$f(u) \geq f(v) + \nabla f(v) \cdot (u - v)$$
2. f is strictly convex on D if and only if for all $u, v \in D$, $u \neq v$, we have that
$$f(u) > f(v) + \nabla f(v) \cdot (u - v)$$

Second order conditions for concavity and convexity.

Proposition

Let $D \subset \mathbb{R}^n$ be open and convex. Let $f \in C^2(D)$ Let $\mathrm{H}\, f(p)$ be the Hessian matrix of f at p. Then,

1. f is concave on D if and only if for every $p \in D$, $\mathrm{H}\, f(p)$ is negative semidefinite or negative definite. That is, f is concave on D if and only if for every $p \in D$ and $x \in \mathbb{R}^n$ we have that $x \cdot \mathrm{H}\, f(p)x \leq 0$.
2. f is convex on D if and only if for every $p \in D$, $\mathrm{H}\, f(p)$ is positive semidefinite or positive definite. That is, f is convex on D if and only if for every $p \in D$ and $x \in \mathbb{R}^n$ we have that $x \cdot \mathrm{H}\, f(p)x \geq 0$.
3. If $\mathrm{H}\, f(p)$ is definite negative for every $p \in D$, then f is strictly concave on D.
4. If $\mathrm{H}\, f(p)$ is positive negative for every $p \in D$, then f is strictly convex on D.

Second order conditions for concavity and convexity.

- If f is strictly convex, then $\mathrm{H}\, f(x, y)$ is positive definite except on a "small" set.
- For example, $f(x, y) = x^4 + y^4$ is strictly convex
- and

$$\mathrm{H}\, f(x, y) = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

  is positive definite if $xy \neq 0$, that is, it is positive definite on all of $\mathbb{R}^2$ except on the two axis $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$.
- For points on the two axis (that is for points $(x, y) \in \mathbb{R}^2$ such that $xy = 0$) the Hessian matrix is positive semidefinite.