Session 12
Mathematics for Economics II

Quadratic forms.

Degrees in Business Administration, Finance and Accounting, Management and Technology, International Studies and Business Administration and Law and Business Administration

Universidad Carlos III de Madrid

Quadratic forms.

- A **quadratic form** of order n is a function $Q : \mathbb{R}^n \to \mathbb{R}$ of the form

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

  for some real numbers $a_{ij} \in \mathbb{R}$ $i, j = 1, \ldots, n$

- $Q(x, y, z) = x^2 - 2xy + 4xz + 6yz + 5z^2$.
- In matrix notation,

$$Q(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= x^2 - 2xy + 4xz + 6yz + 5z^2$$

- There is a unique way if we require that the associated matrix to be symmetric.

- Every quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, can be written in a unique way $Q(x) = xAx^t$ with $A = A^t$ a symmetric matrix.
- Observe that the symmetric matrix

$$A = (a_{ij})$$

is associated with the following quadratic form

$$Q(x) = \sum_{i,j=1}^{n} a_{ij}x_i x_j = \sum_{i=1}^{n} a_{ii}x_i^2 + 2 \sum_{1 \le i < j \le n} a_{ij}x_i x_j$$

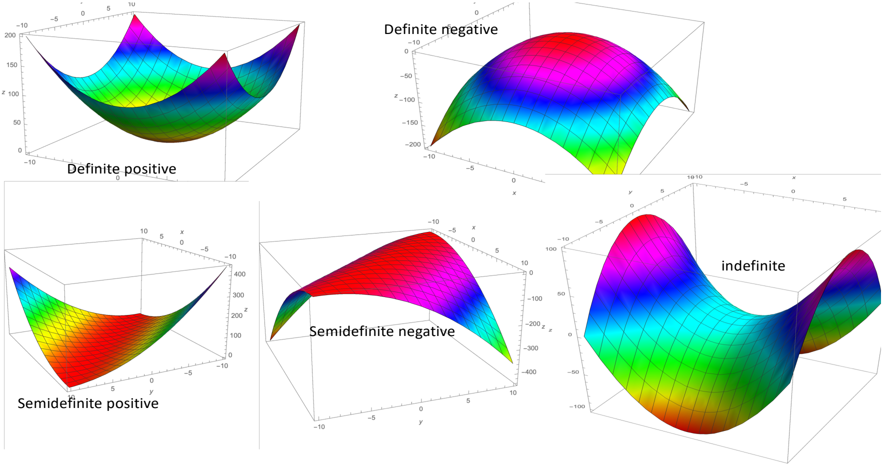
- We will identify the quadratic form $Q(x) = xAx^t$ with the matrix A.

Classification of quadratic forms.

A quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ is

1. **Positive definite** if $Q(x) > 0$ for every $x \in \mathbb{R}^n$, $x \neq 0$.
2. **Negative definite** if $Q(x) < 0$ for every $x \in \mathbb{R}^n$, $x \neq 0$.
3. **Positive semidefinite** if $Q(x) \geq 0$ for every $x \in \mathbb{R}^n$ and $Q(x) = 0$ for some $x \neq 0$.
4. **Negative semidefinite** if $Q(x) \leq 0$ for every $x \in \mathbb{R}^n$ and $Q(x) = 0$ for some $x \neq 0$.
5. **Indefinite** if there are some $x, y \in \mathbb{R}^n$ such that $Q(x) > 0$ and $Q(y) < 0$.

Classification of quadratic forms.



Examples.

- $Q_1(x,y,z) = x^2 + 3y^2 + z^2$ is positive definite.
- $Q_2(x,y,z) = -2x^2 - y^2$ is negative semidefinite.
- $Q_3(x,y) = -2x^2 - y^2$ is negative definite.
- $Q_4(x,y,z) = x^2 - y^2 + 3z^2$ is indefinite.
- The previous quadratic forms are easy to classify because they are in diagonal form, i.e.

$$Q_1 \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad Q_2 \Leftrightarrow \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Q_3 \Leftrightarrow \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \quad Q_4 \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Classification of diagonal quadratic forms.

Proposition

Consider the matrix $A = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$. Then, the quadratic form $Q(x) = xAx^t = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$ is

1. positive definite if and only if $\lambda_i > 0$ for every $i = 1, 2, \ldots, n$;
2. negative definite if and only if $\lambda_i < 0$ for every $i = 1, 2, \ldots, n$;
3. positive semidefinite if and only if $\lambda_i \geq 0$ for every $i = 1, 2, \ldots, n$ and $\lambda_k = 0$ for some $k = 1, 2, \ldots, n$;
4. negative semidefinite if and only if $\lambda_i \leq 0$ for every $i = 1, 2, \ldots, n$ and $\lambda_k = 0$ for some $k = 1, 2, \ldots, n$;
5. indefinite if and only if there is some $\lambda_i > 0$ and some $\lambda_i < 0$.

Leading Principal minors. $|A| \neq 0$

- Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{14} \\ a_{12} & a_{22} & \cdots & a_{24} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$ be a symmetric matrix.
- The leading principal minors are

$$D_1 = a_{11}, \quad D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}, \ \ldots, \ Dn = |A|$$

- Suppose that $|A| \neq 0$. Then,
  1. A is positive definite if and only $D_i > 0$ for every $i = 1, 2, \ldots, n$;
  2. A is negative definite if and only $(-1)^i D_i > 0$ for every $i = 1, 2, \ldots, n$;
  3. if and (1) and (2) do not hold, then Q is indefinite.

Example.

- Consider the quadratic form
  $Q(x, y, z) = x^2 + 2xy - 2xz - 2y^2 + 4yz + 3z^2$.
- The associated matrix is

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 3 \end{pmatrix}$$

- $D_1 = 2 > 0, \; D_2 = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3 < 0.$
- The quadratic form is indefinite.

Example.

- Consider the quadratic form $Q(x, y, z) = 2x^2 + 2xy + y^2 + 2yz + 3z^2$.
- The associated matrix is

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

- $D_1 = 2 > 0,\ D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 > 0,\ D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{vmatrix} = 1 > 0.$
- The quadratic form is positive definite.

Example.

- Consider the quadratic form $Q(x, y, z) = -2x^2 + 2xy - 3y^2 + 2yz - z^2$.
- The associated matrix is

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

- $D_1 = -2 < 0$, $D_2 = \begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} = 5 > 0$,

  $D_3 = \begin{vmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -3 < 0.$
- The quadratic form is negative definite.

Leading Principal minors. $|A| = 0$

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Proposition

Let $Q(x) = xAx^t$ with A symmetric and suppose $D_n = 0$ and $D_1 \neq 0, D_2 \neq 0, \dots, D_{n-1} \neq 0$. Then A is

1. positive semidefinite if and only $D_1, D_2, \dots, D_{n-1} > 0$;
2. negative semidefinite if and only $D_1 < 0, D_2 > 0, \dots, (-1)^{n-1} D_{n-1} > 0$;
3. indefinite otherwise.

- **Example:** Let $Q(x, y) = x^2 + 4xy + 4y^2$. The associated matrix is
$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$
- $D_1 = 1 > 0$, $D_2 = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$.
- The quadratic form is positive semidefinite.

Other cases.

- Consider the matrix
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$$
We see that $D_1 = 1$, $D_2 = D_3 = 0$. Our previous methods do not apply.
- The associated quadratic form is positive semidefinite if and only if $a \geq 0$ and indefinite if and only if $a < 0$.
- If we exchange the variables y and z then the associated matrix becomes
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
and then, the above propositions apply.

Principal centered minors.

- A principal centered minor of order k is the determinant of a submatrix obtained by deleting **same** $n - k$ **rows and columns**.

Principal centered minors. Example

$$A \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{13} \end{pmatrix}$$

1. The 1×1 principal centered minors of A are:
   - a_{11} (obtained by deleting rows and columns 2 and 3).
   - a_{22} (obtained by deleting rows and columns 1 and 3); and
   - a_{33} (obtained by deleting rows and columns 1 and 2).
2. The 2×2 principal centered minors are:
   - $\begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}$, obtained by deleting row and column 1.
   - $\begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix}$, obtained by deleting row and column 2; and
   - $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, obtained by deleting row and column 3.
3. The only 3×3 principal centered minor of A is $|A|$.

Principal minors. Example

The possible chains of leading principal centered minors are:

1. $D_1 = a_{11}$, $D_2 = \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix}$, $D_3 = |A|$.
2. $D_1 = a_{11}$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $D_3 = |A|$.
3. $D_1 = a_{22}$, $D_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}$, $D_3 = |A|$.
4. $D_1 = a_{22}$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $D_3 = |A|$.
5. $D_1 = a_{33}$, $D_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix}$, $D_3 = |A|$.
6. $D_1 = a_{33}$, $D_2 = \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix}$, $D_3 = |A|$.

Proposition

Proposition 12 still holds if we replace the chain of leading principal minors by any other chain consisting of principal centered minors.

Example.

Example

Let $Q(x, y, z) = x^2 - 2xy - 2xz + y^2 + 2yz + 2z^2$

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

We have $D_2 = D_3 = 0$. We consider the chain of principal minors

1. $(1, 2)$, $D_1 = 2$.
2. (1), $D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$.
3. $D_3 = |A| = 0$.

So, the associated quadratic form is positive semidefinite.

Example.

Example

Let $Q(x,y,z) = x^2 - 2xy - 2xz + y^2 + 2yz$. The associated matrix is

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

We have $D_2 = D_3 = 0$. We consider the chain of principal minors

1. $(2,3)$, $D_1 = 1$.
2. (2), $D_2 = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = -1$.
3. $D_3 = |A| = 0$.

So, the associated quadratic form is indefinite.

Remark.

- The methods above are especially useful for symmetric 2×2 matrices.
- For example if A is 2×2 matrix and $|A| < 0$, then the associated quadratic form is indefinite.
- Why?