Session 11
Mathematics for Economics II

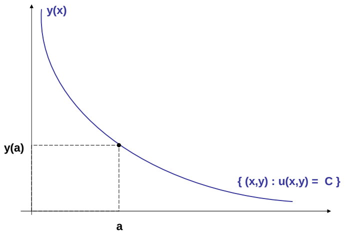
Differentiability. Part VI: Applications of the Implicit Function Theorem. Taylor's Approximations.

Degrees in Business Administration, Finance and Accounting, Management and Technology, International Studies and Business Administration and Law and Business Administration

Universidad Carlos III de Madrid

Indifference curves.

- Two consumption goods and a consumer with a differentiable utility function $u(x, y)$.
- The indifference curves of u are the level sets $\{(x, y) \in \mathbb{R}^2 : x, y > 0, \quad u(x, y) = C\}$.
- Suppose $\frac{\partial u}{\partial x} > 0$ $\frac{\partial u}{\partial y} > 0$.
- By the implicit function Theorem the equation $u(x, y) = C$ defines a function of x.
- The set $\{(x, y) \in \mathbb{R}^2 : x, y > 0, \quad u(x, y) = C\}$ may be represented as the graph of the function $y(x)$.



Indifference curves.

- Differentiating implicitly,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} y'(x) = 0$$

- Hence,

$$y'(x) = -\frac{\partial u / \partial x}{\partial u / \partial y}$$

- $y(x)$ is a decreasing function.
- The marginal rate of substitution is

$$\mathrm{MRS}(x, y) = |y'(x)| = \frac{\partial u / \partial x}{\partial u / \partial y}(x, y)$$

- MRS measures (approximately) the maximum amount of good y that the agent would be willing to exchange for an additional consumption of one unit of good x.

MRS.

- Let $u(x, y) = x^2 y^4$.
- The marginal rate of substitution is

$$\mathrm{MRS}(x, y) = \frac{\partial u / \partial x}{\partial u / \partial y} = \frac{2xy^4}{4x^2y^3} = \frac{y}{2x}$$

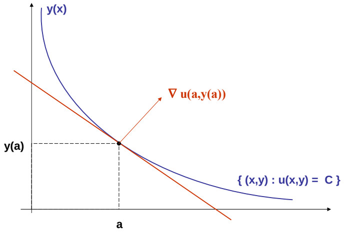
Indifference curves.

- The slope of the straight line tangent to the graph of $y(x)$ at the point $(a, y(a))$ is $y'(a)$
- That is, the director vector of the straight line tangent to the graph of $y(x)$ at the point $(a, y(a))$ is the vector $(1, y'(a))$.
- And

$$(1, y'(a)) \cdot \nabla u(a, y(a)) = \left(1, -\frac{\partial u / \partial x}{\partial u / \partial y}\right) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Indifference curves.

- The gradient vector ∇u is perpendicular to the straight line tangent the indifference curve of the consumer.



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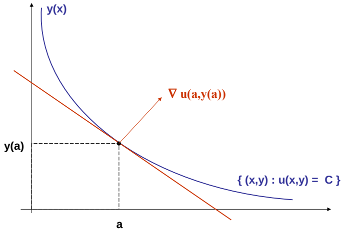

Demand function of a consumer.

- Suppose that there are two consumption goods and the agent has preferences over theses which might be represented by a utility function $u(x, y)$.
- Suppose the prices of the goods are p_x and p_y. Consuming the bundle (x, y) costs
$$p_x x + p_y y$$
to the agent.
- If his income is I then
$$p_x x + p_y y = I$$
- That is, if the agent buys x units of the first good, then the maximum amount he can consume of the second good is
$$\frac{I}{p_y} - \frac{p_x}{p_y} x$$

Demand function of a consumer.

- So, his utility is

$$u\left(x, \frac{I}{p_y} - \frac{p_x}{p_y}x\right) \tag{0.1}$$

- In Economic Theory one assumes that the agent chooses the bundle of goods (x, y) that maximizes his utility.
- That is, the agent maximizes the function $u\left(x, \frac{I}{p_y} - \frac{p_x}{p_y}x\right)$
- Differentiating implicitly with respect to x we obtain

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\frac{p_x}{p_y} = 0 \tag{0.2}$$

Demand function of a consumer.

- Thus, the first order condition is

$$\mathrm{MRS}(x, y) = \frac{p_x}{p_y}$$

The above equation together with the budget restriction

$$p_x x + p_y y = I$$

determines the demand of the agent.

Demand function of a consumer.

- For example if the preferences of the consumer may be represented by a Cobb-Douglas utility function

$$u(x, y) = x^2 y$$

the MRS is

$$\mathrm{MRS}(x, y) = \frac{2xy}{x^2} = \frac{2y}{x}$$

and the demand of the agent is determined by the system of equations

$$\begin{aligned} \frac{2y}{x} &= \frac{p_x}{p_y} \\ p_x x + p_y y &= I \end{aligned}$$

from these we obtain the demand of the agent

$$\begin{aligned} x(p_x, p_y, I) &= \frac{2I}{3p_x} \\ y(p_x, p_y, I) &= \frac{I}{3p_y} \end{aligned}$$

Isoquants and the marginal rate of technical substitution.

- Suppose that a firm uses the production function $Y = f(x_1, x_2)$ where (x_1, x_2) are the units of inputs used in manufacturing of Y units of the product.
- Given a fixed level of production $\bar{y}$, the corresponding isoquant is the level set

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0, \quad f(x_1, x_2) = \bar{y}\}$$

- As in the previous exercise, we see that on the isoquants we may write x_2 as a function of x_1 and that

$$x_2'(x_1) = -\frac{\partial f / \partial x_1}{\partial f / \partial x_2}$$

The marginal rate of technical substitution is defined as

$$\text{RMST} = -x_2'(x_1) = \frac{\partial f / \partial x_1}{\partial f / \partial x_2}$$

Isoquants and the marginal rate of technical substitution.

- For example, if the production function of the firm is $Y = x_1^{1/3} x_2^{1/2}$ then the marginal rate of technical substitution is

$$\text{RMST} = \frac{\partial Y / \partial x_1}{\partial Y / \partial x_2} = \frac{\frac{1}{3} x_1^{-2/3} x_2^{1/2}}{\frac{1}{2} x_1^{1/3} x_2^{-1/2}} = \frac{2x_2}{3x_1}$$

Taylor polynomial of first order.

- Let $f \in C^1(D)$, $p \in D$. The Taylor polynomial of first order at p is

$$P_1(x) = f(p) + \nabla f(p) \cdot (x - p)$$

- If $f(x, y)$ is a function of two variables and $p = (a, b)$, then Taylor's first order polynomial for the function f around the point $p = (a, b)$ is the polynomial

$$P_1(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b)$$

- $\lim_{x \to p} \frac{f(x) - P_1(x)}{\|x - p\|} = 0.$

Taylor polynomial of second order.

- Let $f \in C^2(D)$, $p \in D$. The Taylor polynomial of second order at p is

$$P_2(x) = f(p) + \nabla f(p) \cdot (x-p) + \frac{1}{2}(x-p)\,\mathrm{H}\,f(p)(x-p)$$

- If $f(x,y)$ is a function of two variables and $p=(a,b)$, then Taylor's order polynomial for the function f around the point $p=(a,b)$ is the polynomial

$$P_2(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) +$$
$$+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x \partial x}(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f}{\partial y \partial y}(y-b)^2\right)$$

- $\lim_{x \to p} \frac{f(x)-P_2(x)}{\|x-p\|^2} = 0.$

Example.

- Let $f(x,y) = x^3y - xy + 2x + 2y^2 - 15y + 1$ and $p = (-1,1)$.
- We compute the gradient and the Hessian matrix of the función f at the point p.
- We have

$$\begin{aligned} \nabla f(x,y) &= \left(3x^2y - y + 2, x^3 - x + 4y - 15\right) \\ \mathrm{H}\, f(x,y) &= \begin{pmatrix} 6xy & 3x^2 - 1 \\ 3x^2 - 1 & 4 \end{pmatrix} \end{aligned}$$

- Hence,

$$\begin{aligned} \nabla f(-1,1) &= (4,-11) \\ \mathrm{H}\, f(-1,1) &= \begin{pmatrix} -6 & 2 \\ 2 & 4 \end{pmatrix} \end{aligned}$$

Example.

- Taylor's first order polynomial of the function f at the point p is

$$P_1(x,y) = f(-1,1)+\nabla f(p)\cdot(x+1,y-1) = -14+(4,-11)\cdot(x+1,y-1)$$

$$= -14+4(1+x)-11(-1+y)$$

- Taylor's second order polynomial of the function f at the point p is

$$\begin{aligned} P_2(x,y) &= f(-1,1)+\nabla f(p)\cdot(x+1,y-1)+ \\ &\quad \frac{1}{2}(x+1,y-1)\,\mathrm{H}\,f(-1,1)\begin{pmatrix} x+1 \\ y-1 \end{pmatrix} = \\ &= -14+(4,-11)\cdot(x+1,y-1)+ \\ &\quad \frac{1}{2}\left(-6(x+1)^2+4(x+1)(y-1)+4(y-1)^2\right) \\ &= -3x^2+2xy-4x+2y^2-13y-2 \end{aligned}$$