September 14, 2020

CHAPTER 1: INTRODUCTION TO THE TOPOLOGY OF EUCLIDEAN SPACE

1. Scalar product in $\mathbb{R}^n$

Definition 1.1. Given $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{R}^n$,we define their **scalar product** as

$$x \cdot y = \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

Example 1.2. $(2, 1, 3) \cdot (-1, 0, 2) = -2 + 6 = 4$

Remark 1.3. $x \cdot y = y \cdot x$.

Definition 1.4. Given $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we define its **norm** as

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2}$$

Example 1.5. Example: $\|(-1, 0, 3)\| = \sqrt{10}$

Remark 1.6. Some interpretations of the norm are:

- The norm $\|x\|$ is the distance from x to the origin.
- We may also interpret $\|x\|$ as the length of the vector x.
- The norm $\|x - y\|$ is the distance between x and y.

Remark 1.7. Let θ be the angle between u and v. Then,

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

2. 1. The Euclidean space $\mathbb{R}^n$

Definition 2.1. Given $p \in \mathbb{R}^n$ and $r > 0$ we define the **open ball** of center p and radius r as the set

$$B(p, r) = \{y \in \mathbb{R}^n : \|p - y\| < r\}$$

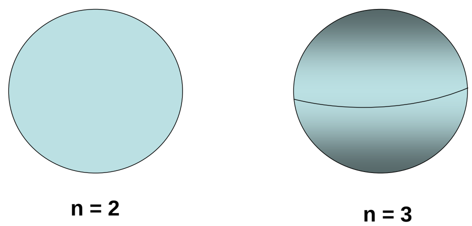
and the **closed ball** of center p and radius r as the set

$$\overline{B(p, r)} = \{y \in \mathbb{R}^n : \|p - y\| \leq r\}$$

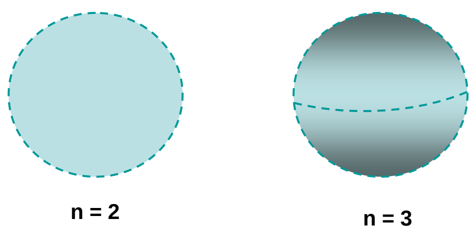
Remark 2.2.

- Recall that $\|p - y\|$ is distance from p to y.
- For $n = 1$, we have that $B(p, r) = (p - r, p + r)$ and $\overline{B(p, r)} = [p - r, p + r]$.

- For $n = 2, 3$ the closed balls are



- For $n = 2, 3$ the open balls are

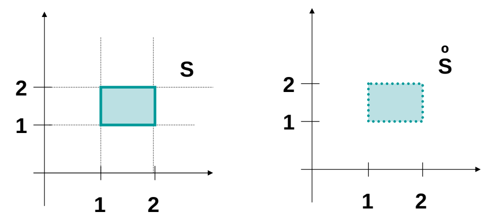


Definition 2.3. Let $S \subset \mathbb{R}^n$. We say that $p \in \mathbb{R}^n$ is **interior** to S if there is some $r > 0$ such that $B(p, r) \subset S$.

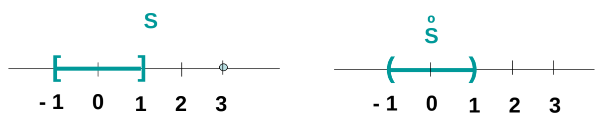
Notation: $\mathring{S}$ is set of interior points of S.

Remark 2.4. Note that $\mathring{S} \subset S$ because $p \in B(p, r)$ for any $r > 0$.

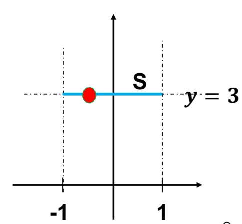
Example 2.5. Consider $S \subset \mathbb{R}^2, S = [1, 2] \times [1, 2]$. Then, $\mathring{S} = (1, 2) \times (1, 2)$.



Example 2.6. Consider $S = [-1, 1] \cup \{3\} \subset \mathbb{R}$. Then, $\overset{\circ}{S} = (-1, 1)$.

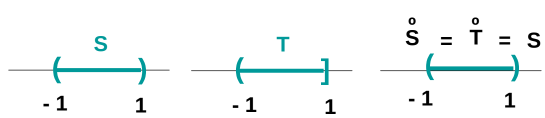


Example 2.7. Consider $S = [-1, 1] \times \{3\} \subset \mathbb{R}^2$. Then, $\overset{\circ}{S} = \emptyset$.

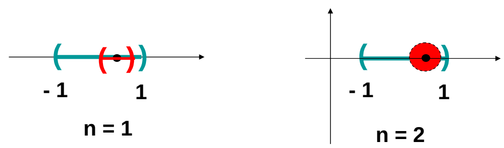


Definition 2.8. A subset $S \subset \mathbb{R}^n$ is **open** if $S = \overset{\circ}{S}$

Example 2.9. In $\mathbb{R}$, the set $S = (-1, 1)$ is open, $T = (-1, 1]$ is not.



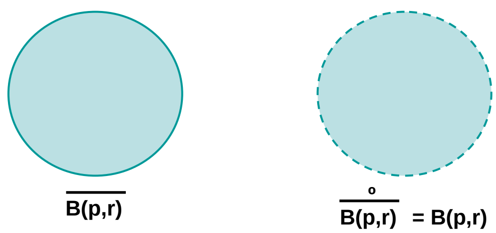
Example 2.10. The set $S = \{(x, 0) : -1 < x < 1\}$ is not open in $\mathbb{R}^2$.



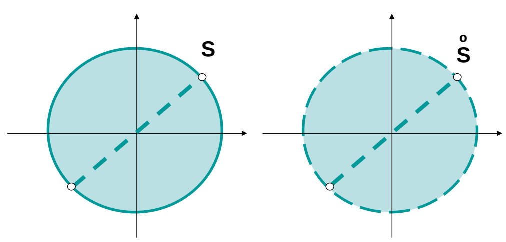
You should compare this with the previous example

Example 2.11. The open ball $B(p, r)$ is an open set.

Example 2.12. The closed ball $\overline{B(p, r)}$ is not an open set, because $\mathring{\overline{B(p, r)}} = B(p, r)$.



Example 2.13. Consider the set $S = \{(x, y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$. Then, $\mathring{S} = \{(x, y) \in R^2 : x^2 + y^2 < 1, x \neq y\}$. So, S is not open.

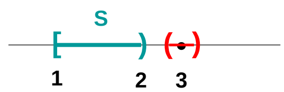


Proposition 2.14. $\mathring{S}$ is the largest open set contained in S. (That is $\mathring{S}$ is open, $\mathring{S} \subset S$ and if $A \subset S$ is open, then $A \subset \mathring{S}$).

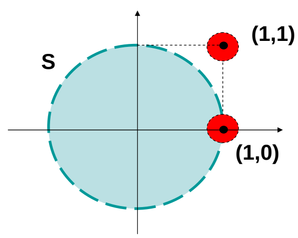
Definition 2.15. Let $S \subset \mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is in the **closure** of S if for any $r > 0$ we have that $B(p, r) \cap S \neq \emptyset$.

Notation: $\bar{S}$ is the set of points in the closure of S.

Example 2.16. Consider the set $S = [1, 2) \subset \mathbb{R}$. Then, the points $1, 2 \in \bar{S}$. But, $3 \notin \bar{S}$.

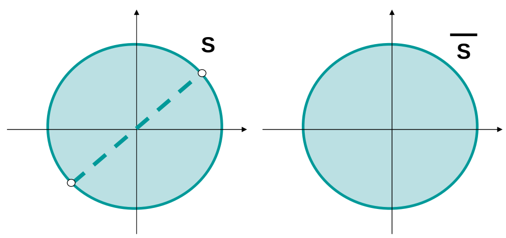


Example 2.17. Consider the set $S = B\left((0,0),1\right) \subset \mathbb{R}^2$. Then, the point $(1,0) \in \bar{S}$. But, the point $(1,1) \notin \bar{S}$.



Example 2.18. Let $S = [0,1]$, $T = (0,1)$. Then, $\bar{S} = \bar{T} = S = [0,1]$.

Example 2.19. Let $S = \{(x,y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$. Then, $\bar{S} = \overline{B\left((0,0),1\right)}$.



Example 2.20. $\overline{B(p,r)}$ is the closure of the open unit ball $B(p,r)$.

Remark 2.21. $S \subset \bar{S}$.

Definition 2.22. A set $F \subset \mathbb{R}^n$ is **closed** if $F = \bar{F}$.

Proposition 2.23. A set $F \subset \mathbb{R}^n$ is closed if and only if $\mathbb{R}^n \setminus F$ is open.

Example 2.24. The set $[1,2] \subset \mathbb{R}$ is closed. But, the set $[1,2) \subset \mathbb{R}$ is not.

Example 2.25. The set $\overline{B(p,r)}$is closed. But, the set $B(p,r)$ is not.

Example 2.26. The set $S = \{(x,y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$is not closed.

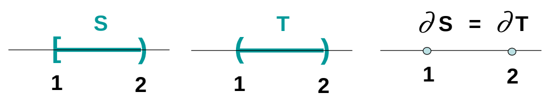
Proposition 2.27. The closure $\bar{S}$ of S is the smallest closed set that contains S. (That is $\bar{S}$ is closed, $S \subset \bar{S}$ and if F is another closed set that contains S, then $\bar{S} \subset F$).

Definition 2.28. Let $S \subset \mathbb{R}^n$, we say that $p \in \mathbb{R}^n$ is a **boundary point** of S if for any positive radius $r > 0$, we have that,

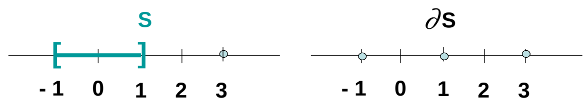
(1) $B(p,r) \cap S \neq \emptyset$.
(2) $B(p,r) \cap (\mathbb{R}^n \setminus S) \neq \emptyset$.

Notation: The set of boundary points of S is denoted by ∂S.

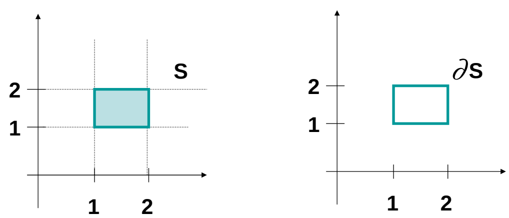
Example 2.29. Let $S = [1, 2)$, $T = (1, 2)$. Then, $\partial S = \partial T = \{1, 2\}$.



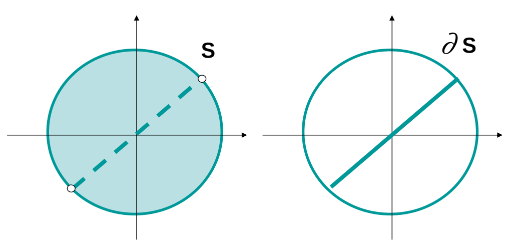
Example 2.30. Let $S = [-1, 1] \cup \{3\} \subset \mathbb{R}$. Then, $\partial S = \{-1, 1, 3\}$.



Example 2.31. Let $S \subset \mathbb{R}^2, S = [1, 2] \times [1, 2]$. Then, ∂S is



Example 2.32. $S = \{(x, y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$. Then, $\partial S = \{(x, y) : x^2 + y^2 = 1\} \bigcup \{(x, y) \in R^2 : x^2 + y^2 \leq 1, x = y\}$.



The above concepts are related in the following Proposition.

Proposition 2.33. Let $S \subset \mathbb{R}^n$, then

(1) $\overset{\circ}{S} = S \setminus \partial S$
(2) $\bar{S} = S \cup \partial S$
(3) $\partial S = \overline{S} \cap \overline{\mathbb{R}^n \setminus S}$.
(4) S is closed $\Leftrightarrow S = \overline{S} \Leftrightarrow \partial S \subset S$
(5) S is open $\Leftrightarrow S = \overset{\circ}{S} \Leftrightarrow S \cap \partial S = \emptyset$.

Proposition 2.34.

(1) The finite intersection of open (closed) sets is also open (closed).
(2) The finite union of open (closed) sets is also open (closed).

Definition 2.35. A set $S \subset \mathbb{R}^n$ is **bounded** if there is some $R > 0$ such that $S \subset B(0, R)$.

Example 2.36. The straight line $V = \{(x, y, z) \in \mathbb{R}^3 : x - y = 0, z = 0\}$ is not a bounded set.

Example 2.37. The ball $B(p, R)$ of center p and radius R is bounded.

Definition 2.38. A subset $S \subset \mathbb{R}^n$ is **compact** if S is closed and bounded.

Example 2.39. $S = \{(x, y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$ is not compact (bounded, but not closed).

Example 2.40. $B(p, R)$ is not compact (bounded, but not closed).

Example 2.41. $\overline{B(p, R)}$ is compact.

Example 2.42. $(0, 1]$ is not compact. $[0, 1]$ is compact.

Example 2.43. $[0, 1] \times [0, 1]$ is compact.

Definition 2.44. A subset $S \subset \mathbb{R}^n$ is **convex** if for any $x, y \in S$ and $\lambda \in [0, 1]$ we have that $\lambda \cdot x + (1 - \lambda) \cdot y \in S$.

Example 2.45. Let A a matrix of order $n \times m$ and let $b \in \mathbb{R}^m$. We define

$$S = \{x \in \mathbb{R}^n : Ax = b\}$$

as the set of solutions of the linear system of equations $Ax = b$. Let $x, y \in S$, be two solutions of this linear system of equations. Then, we have that $Ax = Ay = b$. If we now take any $0 \leq t \leq 1$ (indeed any $t \in \mathbb{R}$) then

$$A(tx + (1 - t)y) = tAx + (1 - t)Ay = tb + (1 - t)b = b$$

that is, $tx + (1 - t)y \in S$ so the set of solutions of a linear system of equations is a convex set.

Example 2.46. $\{(x, y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$ is not a convex set.