University Carlos III
Department of Economics
Mathematics I. Final Exam. January 21st 2022

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Consider the set $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25, x + y \geq 5$.

(a) (**4 points**) Draw the set A, its interior and boundary. Justify if the set A is open, closed, bounded, compact or convex.

(b) (**3 points**) State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f(x,y) = xy$ defined on A. Draw the level curves of $f(x,y) = xy$ definided in $\mathbb{R}^2_+$and the direction of growth of the level curves.

(c) (**3 points**) Using the level curves of f above, determine if the function f attains a global maximum and/or a global minimum on the set A. If so, compute the points where the extreme values are attained and the global maximum and/or minimum value(s) of f on the set A.

(2) Consider the function $f(x,y,z) = 4ax^2 + 4ay^2 + 5xy + 4xz + 2z^2$ defined in $\mathbb{R}^3$, with $a \in \mathbb{R}$.

(a) (**6 points**) Determine for which values of a the function f is strictly convex. Determine for which values of a the function f is strictly concave.

(b) (**4 points**) Using the results above, determine if the set $D = \{(x,y,z) \in \mathbb{R}^3 : 4x^2 + 5xy + 4xz + 4y^2 + 2z^2 \leq 5\}$ is convex.

(3) Consider the set of equations

$$\left.\begin{array}{rcl} xy^2 - yz^2 + yz & = & 1 \\ xe^{2z} - y^2z & = & 1 \end{array}\right\}$$

(a) (**4 points**) Prove that the above set of equations defines implicitly two differentiable functions $y(x)$ and $z(x)$ near the point $(x,y,z) = (1,-1,0)$.

(b) (**6 points**) Compute

$$y'(1), z'(1)$$

and Taylor's polynomial of order one of the function $y(x)$ at the point $x_0 = 1$. Using that polynomial, find and approximation to the value of $y(0.95)$.

(4) Consider the function $f(x,y) = 2x^2y - xy + 2x - 2y^2 - 15y + 1$, the point $p = (1,2)$ and the vector $v = (-1,3)$.

(a) (**5 points**) Compute the gradient and the Hessian matrix of the función f at the point p. Compute $D_v f(p)$.

(b) (**5 points**) Compute the tangent plane to the graph of the function f at the point $(p, f(p))$. Compute Taylor's polynomial of second order of the function f at the point p.

(5) Consider the function $f(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the functions $x(u,v), y(u,v) : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Consider the composition $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $h(u,v) = f(x(u,v), y(u,v))$.

(a) (**2 points**) State the chain rule for the case,

$$\frac{\partial h}{\partial u}, \quad \frac{\partial h}{\partial v}$$

(b) (**5 points**) Use the previous part to compute

$$\frac{\partial h}{\partial u}, \quad \frac{\partial h}{\partial v}$$

for the functions

$$f(x,y) = \frac{2x - y}{x + 3y} \qquad \text{and} \qquad x(u,v) = -ue^{2u}, \quad y(u,v) = v^2e^{2u}$$

at the point $(u_0, v_0) = (0, -1)$.

(c) (**3 points**) Compute the composite function $h(u, v)$, its gradient $\nabla h(u, v)$ and check that $\nabla h(0, -1)$ agrees with the result computed in the preceding part.