University Carlos III
Department of Economics
Mathematics I. Final Exam. January 21st 2022

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

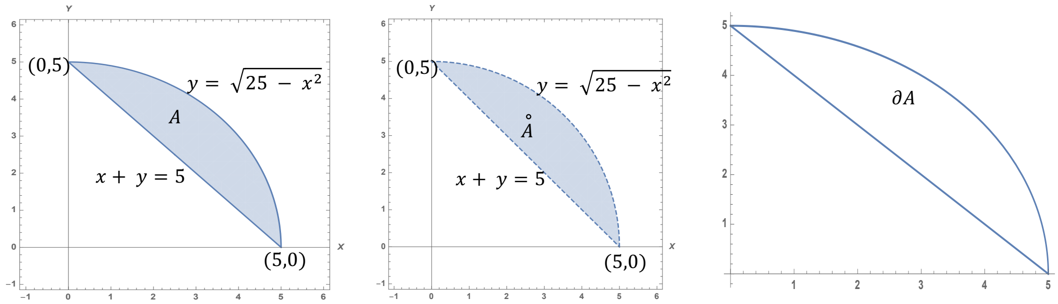
- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Consider the set $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25, x + y \geq 5$.

(a) Draw the set A, its interior and boundary. Justify if the set A is open, closed, bounded, compact or convex.

Solution: *The set A, its interior and its boundary are:*

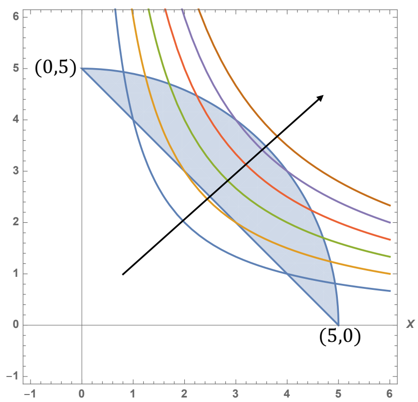


Since, the set A contain its boundary, it is closed. It does not coincide with its interior. Hence, it is not open. Graphically, we see that the set A is bounded and convex. The set A is compact.

(b) State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f(x,y) = xy$ defined on A. Draw the level curves of $f(x,y) = xy$ definided in $\mathbb{R}^2_+$and the direction of growth of the level curves.

Solution: *The function $f(x,y) = xy$ is continuous in $\mathbb{R}^2$. Hence, it is continuous in $A \subset \mathbb{R}^2$. In addition, the set A is compact. Weierstrass' theorem applies.*

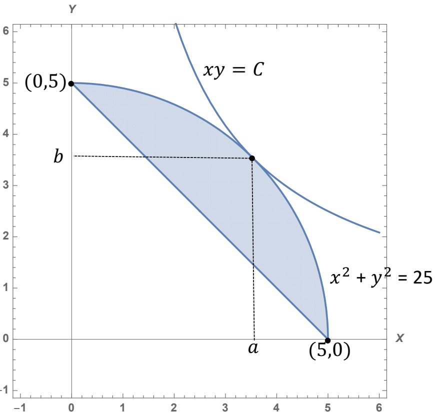
The level curves of the function f are given by the equation $xy = C$. For $C \neq 0$, we obtain $y = \frac{C}{x}$. For $C = 0$, we obtain the coordinate axes. Graphically, for $x, y > 0$,



The arrow points in the direction of growth of C.

(c) Using the level curves of f above, determine if the function f attains a global maximum and/or a global minimum on the set A. If so, compute the points where the extreme values are attained and the global maximum and/or minimum value(s) of f on the set A.

Solution: *Note that on A we have $f(x,y) \geq 0$ and $f(5,0) = f(0,5) = 0$. The global minimum is attained at the points $(5,0)$ and $(0,5)$. Graphically,*



The maximum value on A is attained at the point, say (a, b), of tangency of the curve $xy = C$ and the curve given by the equation $x^2 + y^2 = 25$. Let $y_1(x)$ the function defined by solving for y in the equation $yx = C$. And let $y_2(x)$ the function defined by solving fory in the equation $x^2 + y^2 = 25$. We have that

$$y_1'(a) = y_2'(a)$$

On the other hand, differentiating implicitly the above equations we have that

$$y_1 + xy_1' = 0, \quad \text{and} \quad 2x + 2y_2y_2' = 0$$

We plug in the values $x = a$, $y_1(a) = y_2(a) = b$ and $m = y_1'(a) = y_2'(a)$ and obtain

$$b + am = 0, \quad \text{and} \quad a + bm = 0$$

or $a = -bm = -b^2/a$. Hence, $a^2 = b^2$. Since $a, b \geq 0$, we have $a = b$. And from $a^2 + b^2 = 25$ we obtain

$$a = b = \frac{5}{\sqrt{2}}$$

(2) Consider the function $f(x,y,z) = 4ax^2 + 4ay^2 + 5xy + 4xz + 2z^2$ defined in $\mathbb{R}^3$, with $a \in \mathbb{R}$.

(a) Determine for which values of a the function f is strictly convex. Determine for which values of a the function f is strictly concave.

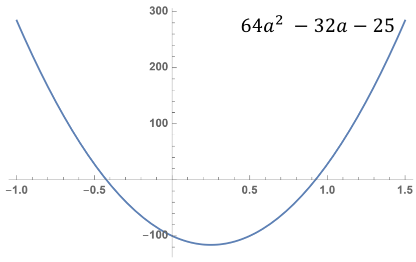
Solution: *We have*

$$\nabla(x,y,z) = (8ax + 5y + 4z, 5x + 8ay, 4x + 4z), \quad \mathrm{H}(f)(x,y,z) = \begin{pmatrix} 8a & 5 & 4 \\ 5 & 8a & 0 \\ 4 & 0 & 4 \end{pmatrix}$$

We consider $D_1 = 4$, $D_2 = 32a$, $D_3 = |A| = 4\left(64a^2 - 32a - 25\right)$. *The roots of* $64a^2 - 32a - 25 = 0$ *are*

$$a = \frac{32 \pm \sqrt{32^2 + 100 \times 64}}{2 \times 64} = \frac{2 \pm \sqrt{29}}{8}$$

Thus, $64a^2 - 32a - 25$ *represents a parabola whose branches point upwards and intersects the* X*-axis at the points* $a_1 = \frac{2-\sqrt{29}}{8} < 0$ *and* $a_2 = \frac{2+\sqrt{29}}{8} > 0$.



(i) *We see that* $D_1 > 0$. *And* $D_2 > 0$ *iff* $a > 0$. *Assuming* $a > 0$, $D_3 > 0$ *iff* $a > \frac{2+\sqrt{29}}{8}$. *We conclude that* $D_1, D_2, D_3 > 0$ *iff* $a > \frac{2+\sqrt{29}}{8}$. *Thus,* f *is strictly convex for* $a > \frac{2+\sqrt{29}}{8}$.

(ii) *Since* $D_1 > 0$, *the function cannot be concave.*

Solution: *We have that*

$$D_1 = 2y, \quad D_2 = 4y - (c + 2x)^2$$

We see that $D_1 > 0$ *if and only if* $y > 0$ *and* $D_2 > 0$ *if and only if* $y > \frac{1}{4}(c+2x)^2$. *Hence,*

$$D = \{(x,y) \in \mathbb{R}^2 : y > \frac{1}{4}(c+2x)^2\}$$

(b) Using the results above, determine if the set $D = \{(x,y,z) \in \mathbb{R}^3 : 4x^2 + 5xy + 4xz + 4y^2 + 2z^2 \leq 5\}$ is convex. **Solution:**
Taking $a = 1 > \frac{2+\sqrt{29}}{8}$ *we have* $f(x,y,z) = ax^2 + 4y^2 + 5xy + 4xz + 2z^2$. *Thus,* $D = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) \leq 5\}$. *Since,* f *is convex, the set* D *is convex.*

(3) Consider the set of equations

$$\left.\begin{array}{rcl} xy^2 - yz^2 + yz & = & 1 \\ xe^{2z} - y^2 z & = & 1 \end{array}\right\}$$

(a) Prove that the above set of equations defines implicitly two differentiable functions $y(x)$ and $z(x)$ near the point $(x, y, z) = (1, -1, 0)$.

Solution: *We check first that* $(x, y, z) = (1, -1, 0)$ *is a solution of the system of equations. The functions* $f_1(x, y, z) = xy^2 - yz^2 + yz - 1$ *and* $f_2(x, y, z) = xe^{2z} - y^2 z - 1$ *contain polynomials and exponentials. Hence, they are of class* C^∞. *We compute the value of*

$$\begin{vmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix}_{(x,y,z)=(1,-1,0)} = \begin{vmatrix} 2xy - z^2 + z & -2yz + y \\ -2yz & 2xe^{2z} - y^2 \end{vmatrix}_{(x,y,z)=(1,-1,0)}$$

$$= \begin{vmatrix} -2 & -1 \\ 0 & 1 \end{vmatrix} = -2 \neq 0$$

By the implicit value theorem, the above sytem of equations determines implicitly two differentiable functions $y(x)$ *and* $z(x)$ *defined near the solution* $(x, y, z) = (1, -1, 0)$.

(b) Compute

$$y'(1), z'(1)$$

and Taylor's polynomial of order one of the function $y(x)$ at the point $x_0 = 1$. Using that polynomial, find and approximation to the value of $y(0.95)$.

Solution: *Differentiating implicitly with respect to* x,

$$\begin{array}{rcl} y^2 + 2xyy' - y'z^2 - 2yzz' + y'z + yz' & = & 0 \\ e^{2z} + 2xz'e^{2z} - 2yy'z - y^2 z' & = & 0 \end{array}$$

Now, we substitute the coordinates of the point $(x, y, z) = (1, -1, 0)$. *We obtain*

$$\begin{array}{rcl} 1 - 2y'(1) - 0 - 0 + 0 - z'(1) & = & 0 \\ 1 + 2z'(1) - z'(1) & = & 0 \end{array}$$

So,

$$z'(1) = -1, \quad y'(1) = 1$$

Therefore Taylor's first order polynomial of the function $y(x)$ *at the point* $x_0 = 1$ *is*

$$P_1(x) = y(1) + y'(1)(x - 1) = -1 + 1(x - 1) = x - 2$$

We use to obtain an approximate value of $y(0.95) \approx P_1(0.95) = 0.95 - 2 = -1.05$

(4) Consider the function $f(x,y) = 2x^2y - xy + 2x - 2y^2 - 15y + 1$, the point $p = (1,2)$ and the vector $v = (-1,3)$.

(a) Compute the gradient and the Hessian matrix of the función f at the point p. Compute $D_v f(p)$.

Solution: *We have*

$$\begin{aligned} \nabla f(x,y) &= \big(4xy - y + 2, 2x^2 - x - 4y - 15\big) \\ \mathrm{H}\, f(x,y) &= \begin{pmatrix} 4y & 4x-1 \\ 4x-1 & -4 \end{pmatrix} \end{aligned}$$

Hence,

$$\begin{aligned} \nabla f(1,2) &= (8,-22) \\ \mathrm{H}\, f(1,2) &= \begin{pmatrix} 8 & 3 \\ 3 & -4 \end{pmatrix} \end{aligned}$$

and

$$D_v f(p) = v \cdot \nabla f(p) = v = (-1,3) \cdot (8,-22) = -74$$

(b) Compute the tangent plane to the graph of the function f at the point $(p, f(p))$. Compute Taylor's polynomial of second order of the function f at the point p.

Solution:
The equation of the tangent plane is

$$\begin{aligned} z &= f(1,2) + \nabla f(p) \cdot (x-1, y-2) = -33 + (8,-22) \cdot (x-1, y-2) = \\ &= -33 + 8(-1+x) - 22(-2+y) \end{aligned}$$

Taylor's second order polynomial of the function f at the point p is

$$\begin{aligned} P_2(x,y) &= f(1,2) + \nabla f(p) \cdot (x-1, y-2) + \frac{1}{2}(x-1, y-2)\, \mathrm{H}\, f(1,2) \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} = \\ &= -33 + \frac{1}{2}((x-1)(8(x-1) + 3(y-2)) + (y-2)(3(x-1) - 4(y-2))) + 8(x-1) - 22(y-2) \\ &= 4x^2 + 3xy - 6x - 2y^2 - 17y + 5 \end{aligned}$$

(5) Consider the function $f(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the functions $x(u,v), y(u,v) : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Consider the composition $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $h(u,v) = f(x(u,v), y(u,v))$.

(a) State the chain rule for the case,

$$\frac{\partial h}{\partial u}, \quad \frac{\partial h}{\partial v}$$

Solution:

$$\frac{\partial h}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial h}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

(b) Use the previous part to compute

$$\frac{\partial h}{\partial u}, \quad \frac{\partial h}{\partial v}$$

for the functions

$$f(x,y) = \frac{2x-y}{x+3y} \qquad \text{and} \qquad x(u,v) = -ue^{2u}, \quad y(u,v) = v^2e^{2u}$$

at the point $(u_0, v_0) = (0, -1)$.

Solution:

$$x(0,-1) = 0, \quad y(0,-1) = 1$$

$$\frac{\partial f}{\partial x} = \frac{7y}{(x+3y)^2} \longrightarrow \frac{\partial f}{\partial x}(0,1) = \frac{7}{9}$$
$$\frac{\partial f}{\partial y} = \frac{-7x}{(x+3y)^2} \longrightarrow \frac{\partial f}{\partial y}(0,1) = 0$$
$$\frac{\partial x}{\partial u} = -e^{2u} - 2ue^{2u} \longrightarrow \frac{\partial x}{\partial u}(0,-1) = -1; \quad \frac{\partial x}{\partial v} = 0 \longrightarrow \frac{\partial x}{\partial v}(0,-1) = 0$$
$$\frac{\partial y}{\partial u} = 2v^2e^{2u} \longrightarrow \frac{\partial y}{\partial u}(0,-1) = 2; \quad \frac{\partial y}{\partial v} = 2ve^{2u} \longrightarrow \frac{\partial y}{\partial v}(0,-1) = -2$$

Hence,

$$\frac{\partial h}{\partial u}(0,-1) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{7}{9}\cdot(-1) + 0\cdot 2 = \frac{-7}{9}$$
$$\frac{\partial h}{\partial v}(0,-1) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = \frac{7}{9}\cdot 0 + 0\cdot(-2) = 0$$

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(c) Compute the composite function $h(u,v)$, its gradient $\nabla h(u,v)$ and check that $\nabla h(0,-1)$ agrees with the result computed in the preceding part.

Solution:

$$h(u,v)\frac{-2ue^{2u} - v^2e^{2u}}{-ue^{2u} + 3v^2e^{2u}} = \frac{2u+v^2}{u-3v^2}$$

$$\vec{\nabla} h(u,v) = \left(\frac{\partial h}{\partial u}, \frac{\partial h}{\partial v}\right) = \left(\frac{2(u-3v^2)-(2u+v^2)}{(u-3v^2)^2}, \frac{2v(u-3v^2)+6v(2u+v^2)}{(u-3v^2)^2}\right) = \left(\frac{-7v^2}{(u-3v^2)^2}, \frac{14uv}{(u-3v^2)^2}\right)$$

and we obtain $\vec{\nabla} h(0,-1) = \left(\frac{-7}{9}, 0\right)$ *which coincides with the previous result.*