University Carlos III
Department of Economics
Mathematics I. Final Exam. January 27th 2021

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Consider the set $A = \{(x,y) \in \mathbb{R}^2 : y^2 \leq x-1, x \leq 5\}$.

(a) (**1 point**) Draw the set A, its interior and boundary. Justify if the set A is open, closed, bounded, compact or convex.

(b) (**1 point**) State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f(x,y) = 2y - x$ defined on A. Draw the level curves of $f(x,y) = 2y - x$ and the direction of growth of the level curves.

(c) (**1 point**) Using the level curves of f above, determine if this function attains a maximum and/or a minimum on the set A. If so, compute the points where the extreme values are attained and the maximum and/or minimum values of f on the set A.

(2) Consider the function $f(x,y) = cxy + x^2y + 2x + y^2 - 15y + 1$ defined in $\mathbb{R}^2$, with $c \in \mathbb{R}$.

(a) (**1 point**) Compute the gradient of f and the Hessian matrix of f. What is the largest open subset D of $\mathbb{R}^2$ in which the function f is strictly convex? (Remark that D depends on c.)

(b) (**1 point**) For what values of c is the set D computed in part (a) convex?

(3) Consider the equation

$$yz - x^2z^3 = 1$$

(a) (**0.5 points**) Using the implicit function theorem prove that the above equation defines a function $z = h(x,y)$ near the point $x = 1$, $y = 2$, $z = 1$.

(b) (**1 point**) Compute

$$\frac{\partial z}{\partial x}(1,2), \quad \frac{\partial z}{\partial y}(1,2),$$

(c) (**0.5 points**) Write the equation of the tangent plane to the graph of the function $z = h(x,y)$, determined in part (a), at the point $q = (1,2)$.

(4) Consider the function

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (**1 point**) Write the definitions of $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$. Compute

$$\frac{\partial f}{\partial x}(0,0), \quad \frac{\partial f}{\partial y}(0,0)$$

(b) (**1 point**) Write the definition that $f(x,y)$ is differentiable at the point $(0,0)$. Prove that the function $f(x,y)$ is differentiable at the point $(0,0)$.

(5) (**1 point**) Let $f(x,y,z) = xy + z^2$, $g(u,v) = (u+v, u-2v, 2u+v)$ and $h(u,v) = f(g(u,v))$. Using the chain rule compute

$$\frac{\partial h}{\partial u}, \quad \frac{\partial h}{\partial v}$$