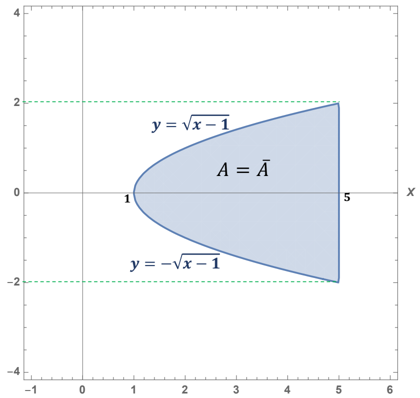
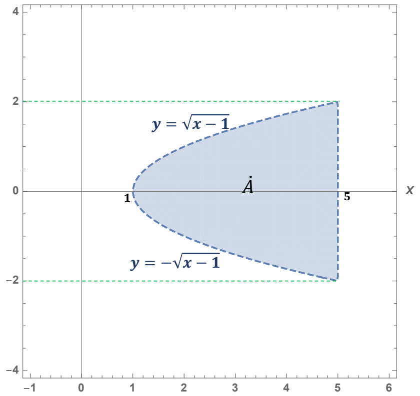
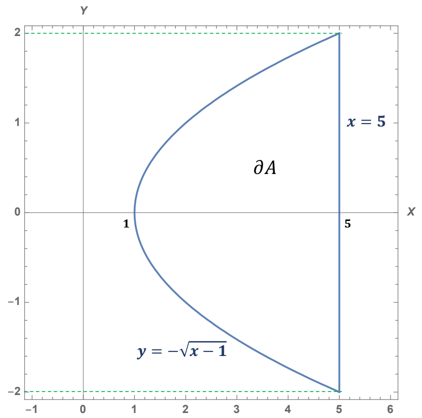
(1) Consider the set $A = \{(x,y) \in \mathbb{R}^2 : y^2 \le x-1, x \le 5\}$.

(a) Draw the set A, its interior and boundary. Justify if the set A is open, closed, bounded, compact or convex.

Solution: *The set A, its interior and its boundary are:*





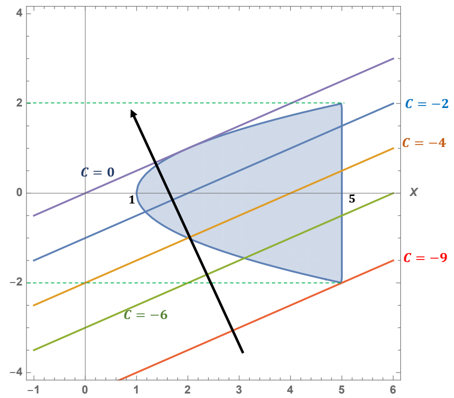


Since, the set A contain its boundary, it is closed. It does not coincide with its interior. Hence, it is not open. Graphically, we see that the set A is bounded and convex. The set A is compact.

(b) State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f(x,y) = 2y - x$ defined on A. Draw the level curves of $f(x,y) = 2y - x$ and the direction of growth of the level curves.

Solution: *The function $f(x,y) = 2y - x$ is continuous in $\mathbb{R}^2$. Hence, it is continuous in $A \subset \mathbb{R}^2$. In addition, the set A is compact. Weierstrass' theorem applies.*

The level curves of the function f are given by the equation $2y - x = C$ or $y = \frac{x}{2} + \frac{C}{2}, \quad C \in \mathbb{R}$. Graphically,



The arrow points in the direction of growth.

(c) Using the level curves of f above, determine if this function attains a maximum and/or a minimum on the set A. If so, compute the points where the extreme values are attained and the maximum and/or minimum values of f on the set A.

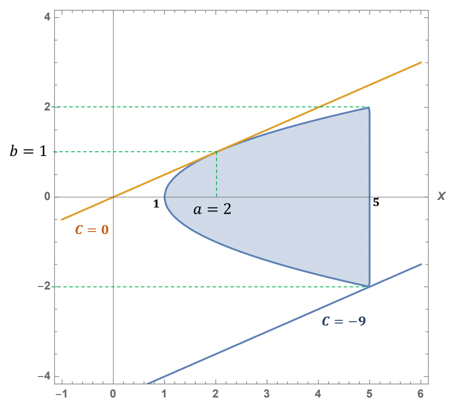
Solution: *Graphically we see that the minimum value of f is attained at the points $(5,-2)$ and $f(5,-2) = -9$. The maximum value is attained at the point, say (a,b), of tangency of the line $2y - x = C$ and the curve given by the equation $y^2 = x - 1$. Differentiating implicitely we have that*

$$2y' - 1 = 0, \quad 2yy' = 1$$

We plug in the values $x = a$, $y = b$ and obtain

$$2y'(a) - 1 = 0, \quad 2by'(a) = 1$$

So, we must have $b = 1$. Since, (a,b) is on the curve $y^2 = x - 1$. We must have $a - 1 = b^2 = 1$. Hence $a = 2$. The maximum is attained at the point $(2,1)$ and $f(2,1) = 0$.



(2) Consider the function $f(x, y) = cxy + x^2y + 2x + y^2 - 15y + 1$ defined in $\mathbb{R}^2$, with $c \in \mathbb{R}$.

(a) Compute the gradient of f and the Hessian matrix of f. What is the largest open subset D of $\mathbb{R}^2$ in which the function f is strictly convex? (Remark that D depends on c.)

Solution: *We have*

$$\nabla f(x, y) = \left(cy + 2xy + 2, cx + x^2 + 2y - 15\right), \quad \mathrm{H}(f)(x, y, z) = \begin{pmatrix} 2y & c + 2x \\ c + 2x & 2 \end{pmatrix}$$

Solution: *We have that*

$$D_1 = 2y, \quad D_2 = 4y - (c + 2x)^2$$

We see that $D_1 > 0$ if and only if $y > 0$ and $D_2 > 0$ if and only if $y > \frac{1}{4}(c + 2x)^2$. Hence,

$$D = \{(x, y) \in \mathbb{R}^2 : y > \frac{1}{4}(c + 2x)^2\}$$

(b) For what values of c is the set D computed in part (b) convex?

Solution: *Consider the function*

$$g(x) = \frac{1}{4}(c + 2x)^2$$

Since $g''(x) = 2 > 0$, the function g is convex. Therefore the set $D = \{(x, y) \in \mathbb{R}^2 : y > \frac{1}{4}(c+2x)^2\}$ is convex for any value of c.

(3) Consider the equation

$$yz - x^2z^3 = 1$$

(a) Using the implicit function theorem prove that the above equation defines a function $z = h(x, y)$ near the point $x = 1$, $y = 2$, $z = 1$.

Solution: *Consider the function* $f(x, y, z) = yz - x^2z^3$. *We see that* $f(1, 2, 1) = 1$. *Furthermore,*

$$\frac{\partial f}{\partial z}(1, 2, 1) = \left(y - 3x^2z^2\right)\big|_{x=1,y=2,z=1} = -1 \neq 0$$

By the implicit function theorem, the equation $f(x, y, z) = 1$ *defines a function* $z = h(x, y)$ *near the point* $(1, 2, 1)$.

(b) Compute

$$\frac{\partial z}{\partial x}(1, 2), \quad \frac{\partial z}{\partial y}(1, 2),$$

Solution: *Differentiating implicitly the equation* $f(x, y, z) = 1$ *we have*

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x} = -3x^2\frac{\partial z}{\partial x}z^2 + y\frac{\partial z}{\partial x} - 2xz^3 \\ 0 &= \frac{\partial f}{\partial y} = -3x^2\frac{\partial z}{\partial y}z^2 + y\frac{\partial z}{\partial y} + z \end{aligned}$$

which is valid for (x, y) *near the point* $(1, 2)$. *Substituting* $x = 1, y = 2, z = 1$ *we have*

$$\begin{aligned} 0 &= -\frac{\partial z}{\partial x}(1, 2) - 2 \\ 0 &= 1 - \frac{\partial z}{\partial y}(1, 2) \end{aligned}$$

And we obtain

$$\frac{\partial z}{\partial x}(1, 2) = -2, \quad \frac{\partial z}{\partial y}(1, 2) = 1$$

(c) Write the equation of the tangent plane to the graph of the function $z = h(x, y)$, determined in part (a), at the point $q = (1, 2)$.

Solution: *The equation of the tangent plane to the graph of the function* $z = h(x, y)$ *at the point* $q = (1, 2)$ *is*

$$z = h(1, 2) + \frac{\partial h}{\partial x}(1, 2)(x - 1) + \frac{\partial h}{\partial y}(1, 2)(y - 2) = 1 - 2(x - 1) + y - 2$$

(4) Consider the function

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Write the definitions of $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$. Compute

$$\frac{\partial f}{\partial x}(0,0), \quad \frac{\partial f}{\partial y}(0,0)$$

Solution: *The definitions are*

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} \\ \frac{\partial f}{\partial y}(0,0) &= \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} \end{aligned}$$

We have that $f(t,0) = f(0,t) = f(0,0) = 0$. *Hence, for* $t \neq 0$,

$$\frac{f(t,0)-f(0,0)}{t} = \frac{f(0,t)-f(0,0)}{t} = 0$$

So,

$$\frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial y}(0,0) = 0$$

(b) Write the definition that $f(x,y)$ is differentiable at the point $(0,0)$. Prove that the function $f(x,y)$ is differentiable at the point $(0,0)$.

Solution:
Let

$$g(x,y) = \frac{f(x,y) - f(0,0) - \nabla f(0,0)\cdot(x-0,y-0)}{\sqrt{x^2+y^2}}$$

The function f *is differentiable at* $(0,0)$ *if*

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

We have that

$$f(0,0) = 0, \quad \nabla f(0,0) = (0,0)$$

Hence,

$$g(x,y) = \frac{x^2y^2}{(x^2+y^2)^{3/2}}$$

And note that

$$0 \leq \frac{x^2y^2}{(x^2+y^2)^{3/2}} \leq \frac{(x^2+y^2)(x^2+y^2)}{(x^2+y^2)^{3/2}} = \left(x^2+y^2\right)^{1/2}$$

Since the function $\left(x^2+y^2\right)^{1/2}$ *is continuous in* $\mathbb{R}^2$ *we have that*

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)^{1/2} = 0$$

Therefore, $\lim_{(x,y)\to(0,0)} g(x,y) = 0$ *and the function* f *is differentiable at the point* $(0,0)$.

(5) Let $f(x, y, z) = xy + z^2$, $g(u, v) = (u + v, u - 2v, 2u + v)$ and $h(u, v) = f(g(u, v))$. Using the chain rule compute

$$\frac{\partial h}{\partial u}, \quad \frac{\partial h}{\partial v}$$

Solution: *We have*

$$Df(x, y, z) = \begin{pmatrix} y & x & 2z \end{pmatrix}$$

Replacing $x = u + v$, $y = u - 2v$, $z = 2u + v$, *we have*

$$Df(u + v, u - 2v, 2u + v) = \begin{pmatrix} u - 2v & u + v & 4u + 2v \end{pmatrix}$$

On the other hand

$$Dg(u, v) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Therefore,

$$Dh(u, v) = \begin{pmatrix} u - 2v & u + v & 4u + 2v \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10u + 3v & 3u - 2v \end{pmatrix}$$

Hence,

$$\frac{\partial h}{\partial u} = 10u + 3v, \quad \frac{\partial h}{\partial v} = 3u - 2v$$