University Carlos III
Department of Economics
Mathematics II. Final Exam. January 14th 2020

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) Consider the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2, xy > 0\}$.

(a) (**0.5 points**) Draw the set A, its interior and boundary. Justify if the set A is open, closed, bounded, compact or convex.

(b) (**0.5 points**) State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f(x, y) = y - x$ defined on A. Draw the level curves of $f(x, y) = y - x$ and the direction of growth of the level curves.

(c) (**1 point**) Using the level curves of f above, determine if this function attains a maximum and/or a minimum on the set A. If so, compute the points where the extreme values are attained and the maximum and/or minimum values of f on the set A.

(2) Consider the function $f(x, y) = ax^2 + by^2 + 2xy + x + y + 1$ in $\mathbb{R}^2$, with $a, b \in \mathbb{R}$.

(a) (**1 point**) Discuss, according to the valores of the parameters a and b, if the function f is strictly concave or strictly convex in $\mathbb{R}^2$.

(b) (**1 point**) Using the above results, determine if the set $A = \{(x, y) \in \mathbb{R}^2 : -x^2 - 4y^2 + 2xy + x + y \geq 6\}$ is convex.

(3) Consider the equation

$$x^2 + y^2 + z^2 + xy + 2z = 1$$

(a) (**0.5 points**) Using the implicit function theorem prove that the above equation defines a function $z = h(x, y)$ near the point $x = 0$, $y = -1$, $z = 0$.

(b) (**1 point**) Compute

$$\frac{\partial z}{\partial x}(0, -1), \quad \frac{\partial z}{\partial y}(0, -1), \quad \frac{\partial^2 z}{\partial x \partial y}(0, -1).$$

(c) (**0.5 points**) Write the equation of the tangent plane to the graph of the function $z = h(x, y)$, computed in part (a), at the point $q = (0, -1)$.

(4) Consider a function $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ and three functions $x(s, t), y(s, t), z(s, t) : \mathbb{R}^2 \to \mathbb{R}$. Consider the composite function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by $h(s, t) = f\left(x(s, t), y(s, t), z(s, t)\right)$.

(a) (**1 point**) State the chain rule for

$$\frac{\partial h}{\partial s}, \quad \frac{\partial h}{\partial t}$$

(b) (**1 point**) Use part (a) to compute

$$\frac{\partial h}{\partial s}, \quad \frac{\partial h}{\partial t}$$

for the functions

$$f(x, y, z) = \frac{1}{2}\left(\ln^2(x) + \ln^2(y) + \ln^2(z)\right)$$

and

$$x(s, t) = e^{(s+t)}, \; y(s, t) = e^{(s-t)}, \; z(s, t) = e^{st}$$

(5) Let $g(x, y) = e^{ax - by}$, $v = (1, -1) \in \mathbb{R}^2$.

(a) (**1 point**) Compute the gradient of g at the point $p = (0, 0)$. Determine for what values of a, b we have that $D_v g(p) = 0$.

(b) (**1 point**) Write the de Taylor polynomial of order 2 of the function $f(x, y) = e^{3x - 2y}$ near the point $p = (0, 0)$.