University Carlos III
Department of Economics
Mathematics II. Final Exam. January 14th 2020

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

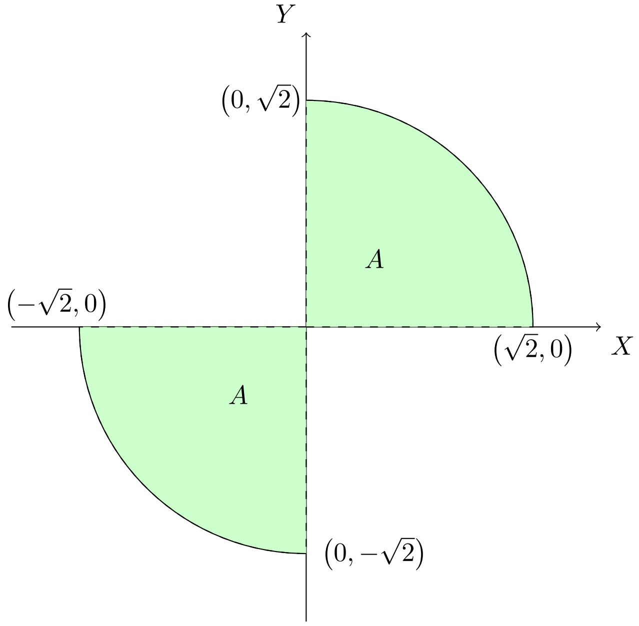
- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

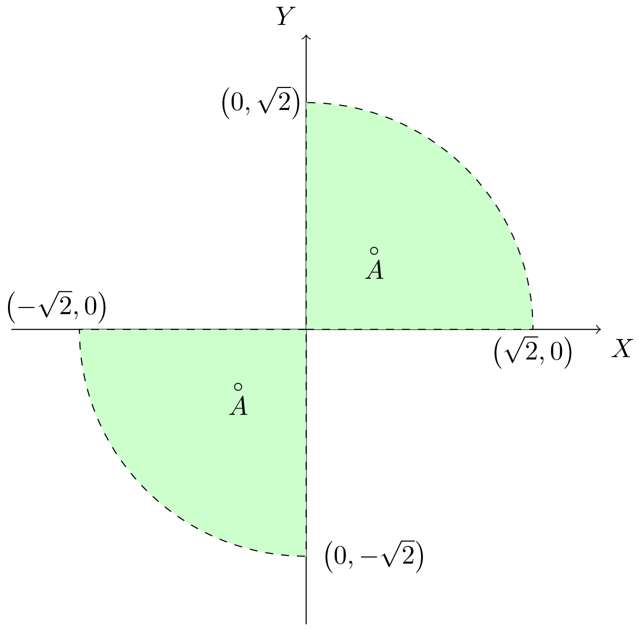
Problem	Points
1	
2	
3	
4	
5	
Total	

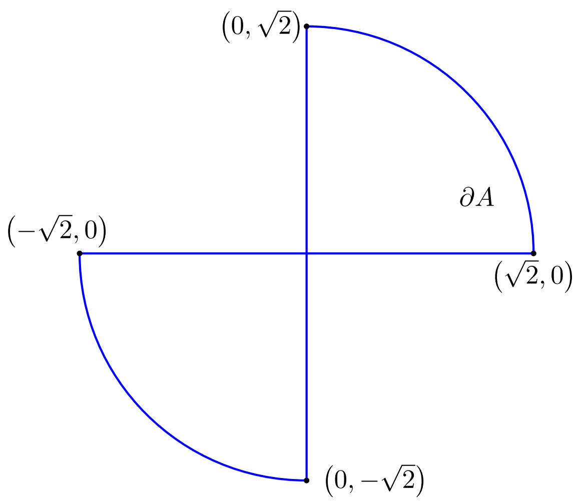
(1) Consider the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2, xy > 0\}$.

(a) Draw the set A, its interior and boundary. Justify if the set A is open, closed, bounded, compact or convex.

Solution: *The set A, its interior and its boundary are:*





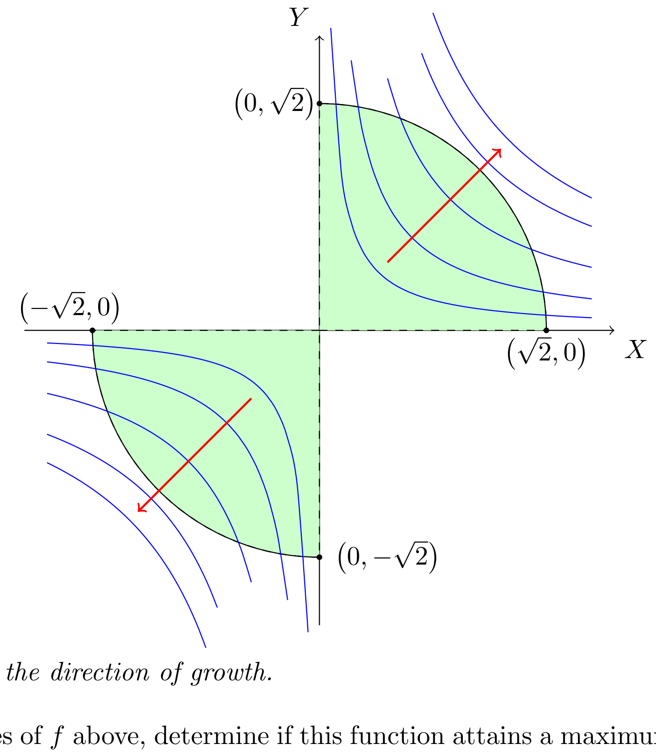


Since, the set A does not contain its boundary, it is not closed. And it does not coincide with its interior. Hence, it is not open. Graphically, we see that te set A is bounded, but not convex. The set A is not compact.

(b) State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f(x,y) = y - x$ defined on A. Draw the level curves of $f(x,y) = y - x$ and the direction of growth of the level curves.

Solution: *The function $f(x,y) = xy$ is continuous in $\mathbb{R}^2$. Hence, it is continuous in $A \subset \mathbb{R}^2$. However, the set A is not compact. The hypotheses of Weierstrass' theorem do not hold.*

The level curves of the function f are given by the equation $y = \frac{C}{x}$, $\quad C \in \mathbb{R}$. Graphically,



The arrows point in the direction of growth.

(c) Using the level curves of f above, determine if this function attains a maximum and/or a minimum on the set A. If so, compute the points where the extreme values are attained and the maximum and/or minimum values of f on the set A.

Solution: *Graphically we see that the maximum value of f is attained at the points where the level curves of the function f are tangent to the boundary of the set A. That is, when the curves*

$$xy = C, \quad x^2 + y^2 = 2$$

intersect at a unique point. Substituting $y = \frac{C}{x}$ *in the second equation, we obtain*

$$x^2 + \frac{C^2}{x^2} = 2$$

that is,

$$x^4 - 2x^2 + C^2 = 0$$

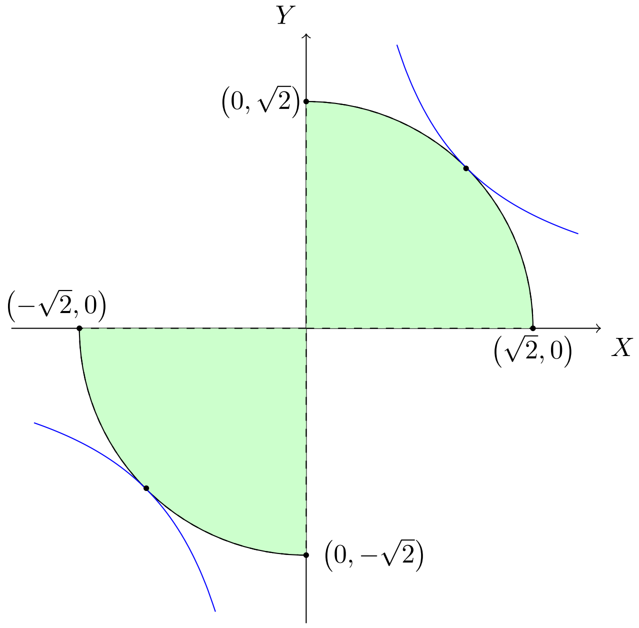
Making the change $t = x^2$*, the above equation reduces to*

$$t^2 - 2t + C^2 = 0$$

The solutions are

$$\frac{2 \pm \sqrt{4 - 4C^2}}{2}$$

There is a unique solution iff $4 - 4C^2 = 0$*, that is* $C^2 = 1$*. We obtain the equation* $t^2 - 2t + 1 = 0$ *whose unique solution is* $t = 1$*. Hence,* $x^2 = 1$ *and from the equation* $x^2 + y^2 = 2$ *we see that* $y^2 = 1$*. Graphically, we see that* x *and* y *have the same sign. The solutions are* $(1, 1)$ *and* $(-1, -1)$*. The maximum value of the function is* $f(1, 1) = f(-1, -1) = 1$*. Graphically,*



Taking points of the form

$$x = y = \frac{1}{n}, \quad con\ n = 1, 2, \ldots$$

we see that

- $\left(\frac{1}{n}, \frac{1}{n}\right) \in A$ *for every* $n = 1, 2, \ldots$
- $\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{n^2}$.

But, there is no point $(x, y) \in A$ *such that* $f(xy) = xy = 0$*, because if* $(x, y) \in A$ *then* $x, y \neq 0$*. Hence, the function* f *does not attain a minimum value in the set* A*.*

(2) Consider the function $f(x, y) = ax^2 + by^2 + 2xy + x + y + 1$ in $\mathbb{R}^2$, with $a, b \in \mathbb{R}$.

(a) Discuss, according to the valores of the parameters a and b, if the function f is strictly concave or strictly convex in $\mathbb{R}^2$.

Solution:
The Hessian matrix of h is

$$\mathrm{H}(h)(x, y, z) = \begin{pmatrix} 2a & 2 \\ 2 & 2b \end{pmatrix}$$

The principal minors are

$$\begin{aligned} D_1 &= 2a \\ D_2 &= 4ab - 4 = 4(ab - 1) \end{aligned}$$

- *If $a > 0$ and $b > 1/a$, then the function f is strictly convex, since $D_1 > 0, D_2 > 0$.*
- *If $a < 0$ and $b < 1/a$, then the function f is concave, since $D_1 < 0, D_2 > 0$.*

(b) Using the above results, determine if the set $A = \{(x, y) \in \mathbb{R}^2 : -x^2 - 4y^2 + 2xy + x + y \geq 6\}$ is convex.

Solution: *Consider the function $g(x, y) = -x^2 - 4y^2 + 2xy + x + y + 1$. This function is obtained from the function $f(x, y) = ax^2 + by^2 + 2xy + x + y + 1$ by taking $a = -1$, $b = -4$. By the previous part, the function g is strictly concave. Since, $A = \{(x, y, z) \in \mathbb{R}^3 : g(x, y) \geq 7\}$, the set A is convex.*

(3) Consider the equation

$$x^2 + y^2 + z^2 + xy + 2z = 1$$

(a) Using the implicit function theorem prove that the above equation defines a function $z = h(x, y)$ near the point $x = 0$, $y = -1$, $z = 0$.

Solution: *Consider the function* $f(x, y, z) = x^2 + y^2 + z^2 + xy + 2$. *We see that* $f(0, -1, 0) = 1$. *Furthermore,*

$$\frac{\partial f}{\partial z}(0, -1, 0) = (2z + 2)|_{x=0,y=-1,z=0} = 2 \neq 0$$

By the implicit function theorem, the equation $f(x, y, z) = 1$ *defines a function* $z = h(x, y)$ *near the point* $(0, -1)$.

(b) Compute

$$\frac{\partial z}{\partial x}(0, -1), \quad \frac{\partial z}{\partial y}(0, -1), \quad \frac{\partial^2 z}{\partial x \partial y}(0, -1).$$

Solution: *Differentiating implicitly the equation* $f(x, y, z) = 1$ *we have*

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} + y + 2\frac{\partial z}{\partial x} \\ 0 &= \frac{\partial f}{\partial y} = 2y + 2z\frac{\partial z}{\partial y} + x + 2\frac{\partial z}{\partial y} \end{aligned}$$

which is valid for (x, y) *near the point* $(0, -1)$. *Substituting* $x = 0, y = -1, z = 0$ *we have*

$$\begin{aligned} 0 &= -1 + 2\frac{\partial z}{\partial x}(0, -1) \\ 0 &= -2 + 2\frac{\partial z}{\partial y}(0, -1) \end{aligned}$$

And we obtain

$$\frac{\partial z}{\partial x}(0, -1) = \frac{1}{2} \quad \frac{\partial z}{\partial y}(0, 1) = 1$$

Differentiating, again, the equation

$$2x + 2z\frac{\partial z}{\partial x} + y + 2\frac{\partial z}{\partial x} = 0$$

with respect to y *we get*

$$2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + 2z\frac{\partial^2 z}{\partial x \partial y} + 1 + 2\frac{\partial^2 z}{\partial x \partial y} = 0$$

Substituting

$$x = 0, y = -1, z = 0, \frac{\partial z}{\partial x}(0, -1) = \frac{1}{2}, \frac{\partial z}{\partial y}(0, 1) = 1$$

we have

$$2\frac{1}{2} + 1 + 2\frac{\partial^2 z}{\partial x \partial y}(0, -1) = 0$$

That is,

$$\frac{\partial^2 z}{\partial x \partial y}(0, -1) = -1$$

(c) Write the equation of the tangent plane to the graph of the function $z = h(x, y)$, computed in part (a), at the point $q = (0, -1)$.

Solution: *The equation of the tangent plane to the graph of the function* $z = h(x, y)$ *at the point* $q = (0, -1)$ *is*

$$z = h(0, -1) + \frac{\partial h}{\partial x}(0, -1)(x - 0) + \frac{\partial h}{\partial y}(0, -1)(y + 1) = 0 + \frac{1}{2}x + y + 1 = \frac{1}{2}x + y + 1$$

Solution: *Differentiating implicitly again the equations*

$$
\begin{aligned}
0 &= \frac{\partial f}{\partial x} = y^2\frac{\partial z}{\partial x} + ye^{xz}\left(x\frac{\partial z}{\partial x} + z\right) \\
0 &= \frac{\partial f}{\partial y} = y^2\frac{\partial z}{\partial y} + xye^{xz}\frac{\partial z}{\partial y} + 2yz + e^{xz}
\end{aligned}
$$

we have

$$
\begin{aligned}
0 &= \frac{\partial^2 f}{\partial x^2} = y\left(e^{xz}\left(x\frac{\partial z}{\partial x} + z\right)^2 + y\frac{\partial^2 z}{\partial x\partial x} + e^{xz}\left(2\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x\partial x}\right)\right) \\
0 &= \frac{\partial^2 f}{\partial x\partial y} = y\left(2\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\partial y}\right) + e^{xz}\left(y\frac{\partial z}{\partial y} + \left(xy\frac{\partial z}{\partial y} + 1\right)\left(x\frac{\partial z}{\partial x} + z\right) + xy\frac{\partial^2 z}{\partial x\partial y}\right) \\
0 &= \frac{\partial^2 f}{\partial y^2} = y\left(4\frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y\partial y}\right) + xe^{xz}\left(\frac{\partial z}{\partial y}\left(xy\frac{\partial z}{\partial y} + 2\right) + y\frac{\partial^2 z}{\partial y\partial y}\right) + 2z
\end{aligned}
$$

Substituting $x = 0, y = 1, z = 2,\ \frac{\partial z}{\partial x}(0,1) = -2,\ \frac{\partial z}{\partial y}(0,1) = -5$ *we have*

$$
\begin{aligned}
0 &= \frac{\partial^2 z}{\partial x^2}(0,1) \\
0 &= \frac{\partial z^2}{\partial x\partial y} - 7 \\
0 &= \frac{\partial^2 z}{\partial y^2} - 16
\end{aligned}
$$

that is,

$$
\frac{\partial^2 z}{\partial x^2}(0,1) = 0, \quad \frac{\partial z^2}{\partial x\partial y} = 7, \quad \frac{\partial^2 z}{\partial y^2} = 16
$$

(4) Consider a function $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ and three functions $x(s, t), y(s, t), z(s, t) : \mathbb{R}^2 \to \mathbb{R}$. Consider the composite function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by $h(s, t) = f(x(s, t), y(s, t), z(s, t))$.

(a) State the chain rule for

$$\frac{\partial h}{\partial s}, \quad \frac{\partial h}{\partial t}$$

Solution:

$$\begin{aligned} \frac{\partial h}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \\ \frac{\partial h}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \end{aligned}$$

(b) Use part (a) to compute

$$\frac{\partial h}{\partial s}, \quad \frac{\partial h}{\partial t}$$

for the functions

$$f(x, y, z) = \frac{1}{2}\left(\ln^2(x) + \ln^2(y) + \ln^2(z)\right)$$

and

$$x(s, t) = e^{(s+t)}, \; y(s, t) = e^{(s-t)}, \; z(s, t) = e^{st}$$

Solution: *By the chain rule*

$$\frac{\partial h}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial h}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

(c)

$$\frac{\partial f}{\partial x} = \frac{\ln(x)}{x} \qquad \frac{\partial f}{\partial y} = \frac{\ln(y)}{y} \qquad \frac{\partial f}{\partial z} = \frac{\ln(z)}{z}$$

$$\begin{aligned} \frac{\partial x}{\partial s} &= e^{(s+t)} = x & \qquad \frac{\partial x}{\partial t} &= e^{(s+t)} = x \\ \frac{\partial y}{\partial s} &= e^{(s-t)} = y & \qquad \frac{\partial y}{\partial t} &= -e^{(s-t)} = -y \\ \frac{\partial z}{\partial s} &= te^{st} = tz & \qquad \frac{\partial z}{\partial t} &= se^{st} = sz \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} &= \frac{\ln(x)}{x}x = \ln(e^{s+t}) = s + t \\ \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} &= \frac{\ln(y)}{y}(y) = \ln(e^{s-t}) = s - t \\ \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} &= \frac{\ln(z)}{z}tz = \ln(e^{st})t = st^2 \\ \frac{\partial h}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} = s + t + s - t + st^2 \end{aligned}$$

$$\boxed{\frac{\partial h}{\partial s} = 2s + st^2}$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} = \frac{\ln(x)}{x}x = \ln(e^{s+t}) = s + t$$

$$\frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\ln(y)}{y}(-y) = -\ln(e^{s-t}) = -s + t$$

$$\frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = \frac{\ln(z)}{z}sz = \ln(e^{st})s = s^2t$$

$$\frac{\partial h}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = s + t - s + t + s^2t$$

$$\boxed{\frac{\partial h}{\partial t} = 2t + s^2t}$$

(5) Let $g(x,y) = e^{ax-by}$, $v = (1,-1) \in \mathbb{R}^2$.

(a) Compute the gradient of g at the point $p = (0,0)$. Determine for what values of a, b we have that $D_v g(p) = 0$.

Solution: *Wr have,*

$$\begin{aligned} \frac{\partial g}{\partial x}(0,0) &= \left(ae^{ax-by}\right)\Big|_{x=0,y=0} = a \\ \frac{\partial g}{\partial y}(0,0) &= \left(-be^{ax-by}\right)\Big|_{x=0,y=0} = -b \end{aligned}$$

Therefore,

$$D_v f(p) = \nabla(p) \cdot v = (a,-b) \cdot (1,-1) = a + b = 0$$

Hence, $D_v f(p) = 0$ if and only if $a = -b$ with $b \in \mathbb{R}$.

(b) Write the de Taylor polynomial of order 2 of the function $f(x,y) = e^{3x-2y}$ near the point $p = (0,0)$.

Solution: *The gradient f is $\nabla f(x,y) = (3e^{3x-2y}, -2e^{3x-2y})$. Therefore,*

$$\nabla f(0,0) = (3,-2)$$

The Hessian matrix of f is

$$\mathrm{H}(f)(x,y) = e^{3x-2y} \begin{pmatrix} 9 & -6 \\ -6 & 4 \end{pmatrix}$$

en el punto $p = (0,0)$,

$$\mathrm{H}(f)(0,0) = \begin{pmatrix} 9 & -6 \\ -6 & 4 \end{pmatrix}$$

Taylor's polynomial is

$$\begin{aligned} P_2(x,y) &= f(0,0) + \nabla f(0,0) \cdot (x,y) + \frac{1}{2} \cdot (x,y) \cdot Hf(0,0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= 1 + (3,-2) \cdot (x,y) + \frac{1}{2} \cdot (x,y) \cdot \begin{pmatrix} 9 & -6 \\ -6 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= 1 + 3x - 2y + \frac{9}{2}x^2 - 6xy + 2y^2 \end{aligned}$$