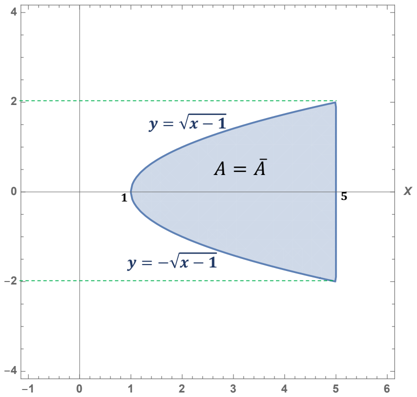
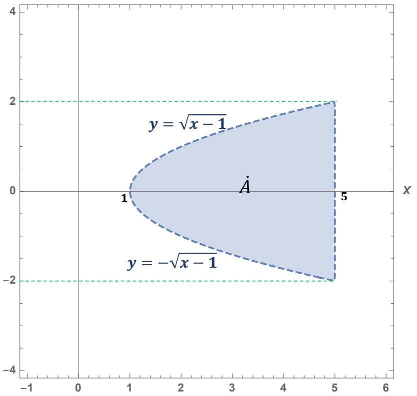
(1) Considere el conjunto $A = \{(x,y) \in \mathbb{R}^2 : y^2 \leq x-1, x \leq 5\}$.

(a) Dibuje el conjunto A, su interior y su frontera. Justifique si el conjunto A es abierto, cerrado, acotado, compacto y/o convexo.

Solución: *The set A, its interior and its boundary are:*



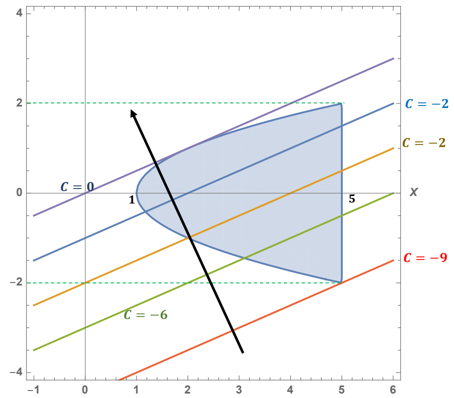


Since, the set A contain its boundary, it is closed. It does not coincide with its interior. Hence, it is not open. Graphically, we see that the set A is bounded and convex. The set A is compact.

(b) Enuncie el teorema de Weierstrass. Determine si es posible aplicar el teorema de Weierstrass a la función $f(x,y) = 2y - x$ definida en A. Dibuje las curvas de nivel de $f(x,y) = 2y - x$ y la dirección de crecimiento de las curvas de nivel.

Solución: *The function $f(x,y) = 2y - x$ is continuous in $\mathbb{R}^2$. Hence, it is continuous in $A \subset \mathbb{R}^2$. In addition, the set A is compact. Weierstrass' theorem applies.*

The level curves of the function f are given by the equation $2y - x = C$ or $y = \frac{x}{2} + \frac{C}{2}, \quad C \in \mathbb{R}$. Graphically,



The arrow points in the direction of growth.

(c) Utilizando las anteriores curvas de nivel de f, determine si esta función alcanza un valor maximo y/o mínimo en el conjunto A. En caso afirmativo, calcule los puntos donde la función f alcanza los valores extremos en el conjunto A.

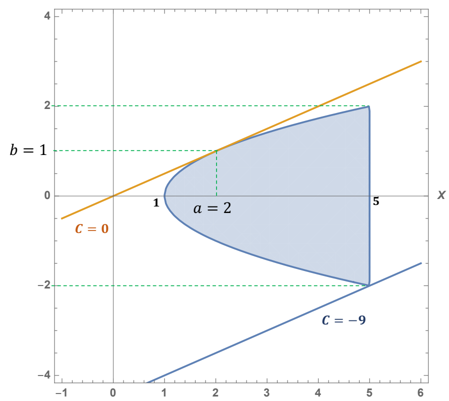
Solución: *Graphically we see that the minimum value of f is attained at the points $(5,-2)$ and $f(5,-2) = -9$. The maximum value is attained at the point, say (a,b), of tangency of the line $2y - x = C$ and the curve given by the equation $y^2 = x - 1$. Differentiating implicitely we have that*

$$2y' - 1 = 0, \quad 2yy' = 1$$

We plug in the values $x = a$, $y = b$ and obtain

$$2y'(a) - 1 = 0, \quad 2by'(a) = 1$$

So, we must have $b = 1$. Since, (a,b) is on the curve $y^2 = x - 1$. We must have $a - 1 = b^2 = 1$. Hence $a = 2$. The maximum is attained at the point $(2,1)$ and $f(2,1) = 0$.



(2) Considere la función $f(x,y) = cxy + x^2y + 2x + y^2 - 15y + 1$ definida en $\mathbb{R}^2$, con $c \in \mathbb{R}$.

(a) Calcule el gradiente de f y la matriz Hessiana de f. ¿Cuál es es el mayor subconjunto abierto D de $\mathbb{R}^2$ en el cual la función f es estrictamente convexa? (El conjunto D depende de c.)

Solución: *We have*

$$\nabla(x,y) = \left(cy + 2xy + 2, cx + x^2 + 2y - 15\right), \quad \mathrm{H}(f)(x,y,z) = \begin{pmatrix} 2y & c + 2x \\ c + 2x & 2 \end{pmatrix}$$

Solución: *We have that*

$$D_1 = 2y, \quad D_2 = 4y - (c + 2x)^2$$

We see that $D_1 > 0$ *if and only if* $y > 0$ *and* $D_2 > 0$ *if and only if* $y > \frac{1}{4}(c+2x)^2$. *Hence,*

$$D = \{(x,y) \in \mathbb{R}^2 : y > \frac{1}{4}(c+2x)^2\}$$

(b) ¿Para qué valores de c es el conjunto D calculado en el apartado (a) convexo?

Solución: *Consider the function*

$$g(x) = \frac{1}{4}(c+2x)^2$$

Since $g''(x) = \frac{1}{2} > 0$, *the function* g *is convex. Therefore the set* $D = \{(x,y) \in \mathbb{R}^2 : y > \frac{1}{4}(c+2x)^2\}$ *is convex for any value of* c.

(3) Considere la ecuación

$$yz - x^2z^3 = 1$$

(a) Utilizando el teorema de la función implicíta, demuestre que la ecuación define una función $z = h(x,y)$ cerca del punto $x = 1$, $y = 2$, $z = 1$.

Solución: *Consider the function* $f(x,y,z) = yz - x^2z^3$. *We see that* $f(1,2,1) = 1$. *Furthermore,*

$$\frac{\partial f}{\partial z}(1,2,1) = \left(y - 3x^2z^2\right)\big|_{x=1,y=2,z=1} = -1 \neq 0$$

By the implicit function theorem, the equation $f(x,y,z) = 1$ *defines a function* $z = h(x,y)$ *near the point* $(1,2)$.

(b) Calcule

$$\frac{\partial z}{\partial x}(1,2), \quad \frac{\partial z}{\partial y}(1,2),$$

Solución: *Differentiating implicitly the equation* $f(x,y,z) = 1$ *we have*

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x} = -3x^2\frac{\partial z}{\partial x}z^2 + y\frac{\partial z}{\partial x} - 2xz^3 \\ 0 &= \frac{\partial f}{\partial y} = -3x^2\frac{\partial z}{\partial y}z^2 + y\frac{\partial z}{\partial y} + z \end{aligned}$$

which is valid for (x,y) *near the point* $(1,2)$. *Substituting* $x = 1, y = 2, z = 1$ *we have*

$$\begin{aligned} 0 &= -\frac{\partial z}{\partial x}(1,2) - 2 \\ 0 &= 1 - \frac{\partial z}{\partial y}(1,2) \end{aligned}$$

And we obtain

$$\frac{\partial z}{\partial x}(1,2) = -2 \quad \frac{\partial z}{\partial y}(0,1) = 1$$

(c) Escriba la ecuación del plano tangente a la gráfica de la función $z = h(x,y)$, determinada en el apartado (a), en el punto $q = (1,2)$.

Solución: *The equation of the tangent plane to the graph of the function* $z = h(x,y)$ *at the point* $q = (1,2)$ *is*

$$z = h(1,2) + \frac{\partial h}{\partial x}(1,2)(x-1) + \frac{\partial h}{\partial y}(1,2)(y-2) = -2(x-1) + y - 1$$

(4) Considere la función

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases}$$

(a) Escriba las definiciones de $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$. Calcule

$$\frac{\partial f}{\partial x}(0,0), \quad \frac{\partial f}{\partial y}(0,0)$$

Solución: *The definitions are*

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} \\ \frac{\partial f}{\partial y}(0,0) &= \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} \end{aligned}$$

We have that $f(t,0) = f(0,t) = f(0,0) = 0$. *Hence, for* $t \neq 0$,

$$\frac{f(t,0)-f(0,0)}{t} = \frac{f(0,t)-f(0,0)}{t} = 0$$

So,

$$\frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial y}(0,0) = 0$$

(b) Escriba la definición de que la función $f(x,y)$ es diferenciable en el punto $(0,0)$. Demuestre que la función $f(x,y)$ es diferenciable en el punto $(0,0)$.

Solución:
Let

$$g(x,y) = \frac{f(x,y) - f(0,0) - \nabla f(0,0)\cdot(x-0, y-0)}{\sqrt{x^2+y^2}}$$

The function f *is differentiable at* $(0,0)$ *if*

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

We have that

$$f(0,0) = 0, \quad \nabla f(0,0) = (0,0)$$

Hence,

$$g(x,y) = \frac{x^2y^2}{(x^2+y^2)^{3/2}}$$

And note that

$$0 \le \frac{x^2y^2}{(x^2+y^2)^{3/2}} \le \frac{(x^2+y^2)(x^2+y^2)}{(x^2+y^2)^{3/2}} = \left(x^2+y^2\right)^{1/2}$$

Since the function $\left(x^2+y^2\right)^{1/2}$ *is continuous in* $\mathbb{R}^2$ *we have that*

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)^{1/2} = 0$$

Therefore, $\lim_{(x,y)\to(0,0)} g(x,y) = 0$ *and the function* f *is differentiable at the point* $(0,0)$.

(5) Sea $f(x,y,z) = xy + z^2$, $g(u,v) = (u+v, u-2v, 2u+v)$ y $h(u,v) = f(g(u,v))$. Utilizando la regla de la cadena calcule

$$\frac{\partial h}{\partial u}, \quad \frac{\partial h}{\partial v}$$

Solución: *We have*

$$Df(x,y,z) = \begin{pmatrix} y & x & 2z \end{pmatrix}$$

Replacing $x = u+v$, $y = u-2v$, $z = 2u+v$, *we have*

$$Df(u+v, u-2v, 2u+v) = \begin{pmatrix} u-2v & u+v & 4u+2v \end{pmatrix}$$

On the other hand

$$Dg(u,v) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Therefore,

$$Dh(u,v) = \begin{pmatrix} u-2v & u+v & 4u+2v \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10u+3v \\ 3u-2v \end{pmatrix}$$

Hence,

$$\frac{\partial h}{\partial u} = 10u + 3v, \quad \frac{\partial h}{\partial v} = 3u - 2v$$