(1) Consider the set

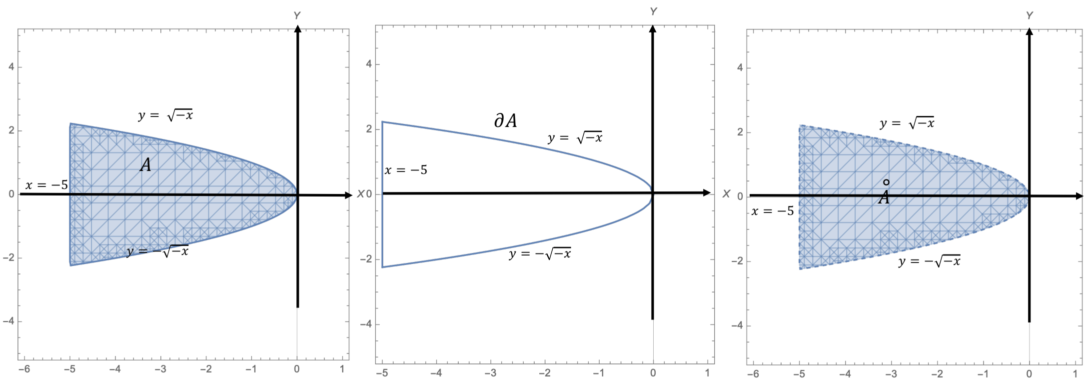
$$A = \{(x, y) \in \mathbb{R}^2 : y^2 + x \leq 0 \leq x + 5\}$$

and the function

$$f(x, y) = (4y + x)^5$$

(a) **(10 points)** Sketch the graph of the set $A$, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.

**Solution:** *The set $A$, its interior and its boundary are approximately as indicated in the picture.*




*And the closure coincides with $A$, $\bar{A} = A$. The functions $h(x, y) = y^2 + x$ and $g(x, y) = x + 5$ are continuous and $A = \{(x, y) \in \mathbb{R}^2 : h(x, y) \leq 0, g(x, y) \geq 0\}$. Hence, the set $A$ is closed (Note also that $\partial A \subset A$). It is not open because $A \cap \partial A \neq \emptyset$. The set $A$ is bounded. Therefore, the set $A$ is compact. It is also convex .*

(b) **(5 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f$ defined on $A$.

**Solution:** *The set $A$ is compact and the function $f(x, y) = (4y + x)^5$ is continuous. Hence, Weierstrass Theorem applies.*

(c) **(5 points)** Draw the level curves of $f$, indicating the direction of growth of the function.

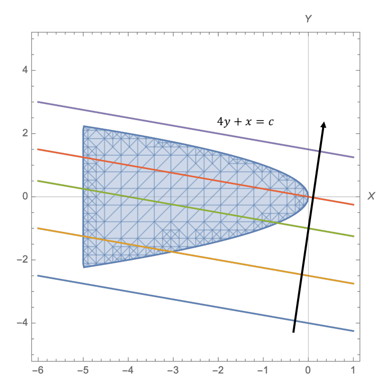
**Solution:** *The level curves*

$$f(x, y) = (4y + x)^5 = D$$

*are straight lines of the form*

$$y = D^{1/5} - \frac{x}{4}$$

*Graphically,*



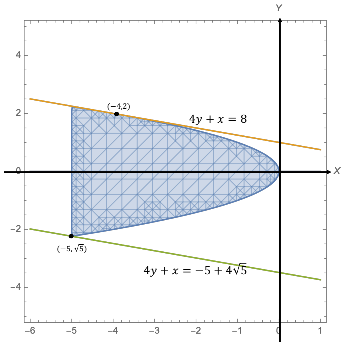

*The black arrow represents the direction of growth of the function $f$.*

(d) **(10 points)** Using the level curves of $f$, determine (if it exists) the **global minimum** and **maximum** of $f$ on the set $A$.

**Solution:** *Graphically, the maximum value is attained at the point* $(x_0, f(x_0))$ *where the line* $y = C - \frac{x}{4}$ *is tangent to the graph of the function* $g(x) = \sqrt{-x}$. *The slope of the line* $y = C - \frac{x}{4}$ *is* $m = -\frac{1}{4}$. *Thus*

$$g'(x_0) = \frac{-1}{2\sqrt{-x_0}} = -\frac{1}{4}$$

*Therefore,* $x_0 = -4$, $y_0 = \sqrt{-x_0} = 2$. *The maximum value is attained at the point* $(-4, 2)$ *and the maximum value of the function is* $f(-4, 2) = (4)^5$.*Graphically,*




*The minimum value is attained at the point where the curves* $y = -\sqrt{-x}$ *and* $x = -5$ *intersect. That is at the point* $(-5, -\sqrt{5})$. *The minimum value is* $f(-5, -\sqrt{5}) = (4\sqrt{5} - 5)^5$.

(2) Consider the function $f(x,y,z) = 2ax^2 + 4axy + 3ay^2 + byz + cz^2 + 13x - 20y + z$ defined in $\mathbb{R}^3$, with $a,b,c \in \mathbb{R}$ and $a \neq 0$.

(a) **(8 points)** Determine for which values of $a,b,c$ the function $f$ is strictly convex. Determine for which values of $a,b,c$ the function $f$ is strictly concave.

**Solution:** *We have*

$$\nabla f(x,y,z) = (4ax + 4ay + 13, 4ax + 6ay + bz - 20, by + 2cz + 1\}), \quad \mathrm{H}(f)(x,y,z) = \begin{pmatrix} 4a & 4a & 0 \\ 4a & 6a & b \\ 0 & b & 2c \end{pmatrix}$$

*We obtain* $D_1 = 4a$, $D_2 = 8a^2 > 0$, $D_3 = |A| = 16a^2c - 4ab^2 = 4a\left(4ac - b^2\right)$. *We see that for* $a > 0$ *and* $c > \frac{b^2}{4a}$ *we have that* $D_1, D_2, D_3 > 0$ *and the function is convex. For* $a < 0$ *and* $c < \frac{b^2}{4a}$ *we have that* $D_1 < 0, D_2 > 0, D_3 < 0$ *and the function is concave.*

(b) **(2 points)** Using the results above, determine if the set $D = \{(x,y,z) \in \mathbb{R}^3 : -2x^2 - 4xy + 13x - 3y^2 + yz - 20y - z^2 + z \geq 10\}$ is convex.

**Solution:**

*Taking* $a = -1$, $b = 1$, $c = -1$ *we obtain the function* $f(x,y,z) = -2x^2 - 4xy + 13x - 3y^2 + yz - 20y - z^2 + z$. *Since* $a < 0$ *and* $c < \frac{b^2}{4a}$, *the function* $f$ *is concave and the set* $D$ *is convex.*

(3) Consider the system of equations

$$\begin{aligned} 2xy + z^2 &= 1 \\ x + y^2 + z &= 0 \end{aligned}$$

(a) **(5 points)** Using the implicit function theorem, prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.

**Solution:** *We first remark that $(x_0, y_0, z_0) = (1, 0, -1)$ is a solution of the system of equations. The functions $f_1(x, y, z) = 2xy + z^2$ and $f_2(x, y, z) = x + y^2 + z$ are of class $C^\infty$, because they are polynomials. We compute*

$$\left| \frac{\partial(f_1, f_2)}{\partial(y, z)} \right|_{(x,y,z)=(1,0,-1)} = \begin{vmatrix} 2x & 2xz \\ 2xy & 1 \end{vmatrix}_{(x,y,z)=(1,0,-1)} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} = 2$$

*By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.*

(b) **(10 points)** Compute

$$y'(1), z'(1)$$

**Solution:** *Differentiating implicitly with respect to $x$,*

$$\begin{aligned} 0 &= 2xy'(x) + 2y(x) + 2z(x)z'(x) \\ 0 &= 2y(x)y'(x) + z'(x) + 1 \end{aligned}$$

*We plug in the values $(x_0, y_0, z_0) = (1, 0, -1)$ to obtain*

$$\begin{aligned} 0 &= 2y'(1) - 2z'(1) \\ 0 &= z'(1) + 1 \end{aligned}$$

*Therefore*

$$y'(1) = -1, \quad z'(1) = -1$$

(c) **(5 points)** Compute Taylor's polynomial of order 1 of the functions $y(x)$ and $z(x)$ at the point $x_0 = 1$.

**Solution:** *Taylor's polynomial of order 1 of the functions $y(x)$ at the point $x_0 = 1$ is*

$$P_1(x) = y(x_0) + y'(x_0)(x - x_0) = 1 - x$$

*Taylor's polynomial of order 1 of the function $z(x)$ at the point $x_0 = 1$ is*

$$P_1(x) = z(x_0) + z'(x_0)(x - x_0) = -x$$

(d) **(5 points)** Compute Taylor's polynomial of order 2 of the functions $y(x)$ and $z(x)$ at the point $x_0 = 1$.

**Solution:** *Differentiating implicitly with respect to $x$ the following system of equations*

$$\begin{aligned} 0 &= 2xy'(x) + 2y(x) + 2z(x)z'(x) \\ 0 &= 2y(x)y'(x) + z'(x) + 1 \end{aligned}$$

*we obtain*

$$\begin{aligned} 0 &= 2xy''(x) + 4y'(x) + 2z(x)z''(x) + 2z'(x)^2 \\ 0 &= 2y(x)y''(x) + 2y'(x)^2 + z''(x) \end{aligned}$$

*We plug in the values $(x_0, y_0, z_0) = (1, 0, -1)$, $y'(1) = z'(1) = -1$ to obtain*

$$\begin{aligned} 0 &= 2y''(1) - 2z''(1) - 2 \\ 0 &= z''(1) + 2 \end{aligned}$$

*Hence, $z''(1) = -2$ and $y''(1) = -1$. Taylor's polynomial of order 2 of the function $y(x)$ at the point $x_0 = 1$ is*

$$P_2(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2}(x - x_0)^2 = 1 - x - \frac{1}{2}(x - 1)^2$$

*Taylor's polynomial of order 2 of the function $z(x)$ at the point $x_0 = 1$ is*

$$P_2(x) = z(x_0) + z'(x_0)(x - x_0) + \frac{z''(x_0)}{2}(x - x_0)^2 = -x - (x - 1)^2$$

(4) Consider the function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) **(5 points)** Is the function $f$ continuous at $(0,0)$?

**Solution:** *We have*

$$0 \leq |f(x,y) - f(0,0)| = \left| \frac{xy^2}{x^2+y^2} \right| \leq |x|$$

*By the squeeze Theorem,* $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$. *Hence,* $f$ *is continuous at* $(0,0)$.

(b) **(5 points)** Compute $\nabla f(0,0)$.

**Solution:** $\nabla f(0,0) = (0,0)$.

(c) **(5 points)** Is the function $f$ differentiable at $(0,0)$?

**Solution:**

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (0,0)\cdot(x,y)}{\sqrt{x^2+y^2}} &= \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} \\ &= \lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}} \end{aligned}$$

*We prove that the above limit does not exist. Consider the function*

$$g(x,y) = \frac{xy^2}{(x^2+y^2)^{3/2}}$$

*Note that*

$$\lim_{t\to 0} g(t,0) = \lim_{t\to 0} \frac{0}{(2t^2)^{3/2}} = 0$$

*and note that*

$$\lim_{t\to 0} g(t,t) = \lim_{t\to 0} \frac{t^3}{(2t^2)^{3/2}} = \frac{1}{(2)^{3/2}} \neq 0$$

*so the limit*

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}}$$

*does not exist and we conclude that* $f$ *is not differentiable at the point* $(0,0)$.

(5) **(10 points)** Consider the function $f(u,v) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the functions $u(x,y,z), v(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$f(u,v) = u^2 + uv \qquad \text{and} \qquad u(x,y,z) = e^x + y^2 + z, \quad v(x,y,z) = x^2 + e^{y^2} + \ln(z)$$

And consider the composition $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $h(x,y,z) = f(u(x,y,z), v(x,y,z))$. Use the the chain rule to compute

$$\frac{\partial h}{\partial x}(0,0,1), \quad \frac{\partial h}{\partial y}(0,0,1), \quad \frac{\partial h}{\partial z}(0,0,1)$$

**Solution:**

$$u(0,0,1) = 2, \quad v(0,0,1) = 1$$

$$Df(u,v) = (2u+v, u), \quad Df(2,1) = (5,2)$$

*Let* $g(x,y,z) = (u(x,y,z), v(x,y,z)) = (e^x + y^2 + z, x^2 + e^{y^2} + \ln(z))$. *We have*

$$Dg(x,y,z) = \begin{pmatrix} e^x & 2y & 1 \\ 2x & 2e^{y^2}y & \frac{1}{z} \end{pmatrix}, \quad Dg(0,0,1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

*By the chain rule,*

$$\begin{pmatrix} \frac{\partial h}{\partial u}(0,0,1) & \frac{\partial h}{\partial v}(0,0,1) \end{pmatrix} = Df(2,1)\, Dg(0,0,1) = (5,2) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = (5,0,7)$$

*Therefore,*

$$\frac{\partial h}{\partial x}(0,0,1) = 5, \quad \frac{\partial h}{\partial y}(0,0,1) = 0, \quad \frac{\partial h}{\partial z}(0,0,1) = 7$$

(6) Consider the surface given by the equation

$$x^2y - 5xyz + 2yz = 16$$

(a) **(5 points)** Write the equation of the tangent plane to the surface at the point $p = (-1, 2, 1)$.

**Solution:** *Consider the funcion* $f(x, y, z) = x^2y - 5xyz + 2yz$. *It is a* $C^\infty$ *function. The gradient of the function* $f$ *is* $\nabla f(x, y, z) = \big(2xy - 5yz, x^2 - 5xz + 2z, 2y - 5xy\big)$. *At the point* $p = (-1, 2, 1)$ *we get* $\nabla f(-1, 2, 1) = (-14, 8, 14\})$. *The equation of the tangent plane is*

$$-14(x + 1) + 8(y - 2) + 14(z - 1) = 0$$

*or* $14x - 8y - 14z = -44$.

(b) **(5 points)** Write the parametric equations of the normal line to the surface at the point $p = (-1, 2, 1)$.

**Solution:** *From the previous part we have that the parametric equations of the normal line are*

$$(x, y, z) = (-1, 2, 1) + t(-14, 8, 14)$$

*That is,*

$$x = -1 - 14t, \quad y = 2 + 8t, \quad z = 1 + 14t$$