University Carlos III
Mathematics for Economics I.
Final Exam. January 19th 2024

Last Name:	**Name:**
Degree:	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
6	
Total	

(1) Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : y^2 + x \leq 0 \leq x + 5\}$$

and the function

$$f(x, y) = (4y + x)^5$$

(a) **(10 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.
(b) **(5 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A.
(c) **(5 points)** Draw the level curves of f, indicating the direction of growth of the function.
(d) **(10 points)** Using the level curves of f, determine (if it exists) the **global minimum** and **maximum** of f on the set A.

(2) Consider the function $f(x, y, z) = 2ax^2 + 4axy + 3ay^2 + byz + cz^2 + 13x - 20y + z$ defined in $\mathbb{R}^3$, with $a, b, c \in \mathbb{R}$ and $a \neq 0$.
(a) **(8 points)** Determine for which values of a, b, c the function f is strictly convex. Determine for which values of a, b, c the function f is strictly concave.

(b) **(2 points)** Using the results above, determine if the set $D = \{(x, y, z) \in \mathbb{R}^3 : -2x^2 - 4xy + 13x - 3y^2 + yz - 20y - z^2 + z \geq 10\}$ is convex.

(3) Consider the system of equations

$$\begin{aligned} 2xy + z^2 &= 1 \\ x + y^2 + z &= 0 \end{aligned}$$

(a) **(5 points)** Using the implicit function theorem, prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.

(b) **(10 points)** Compute

$$y'(1), z'(1)$$

(c) **(5 points)** Compute Taylor's polynomial of order 1 of the functions $y(x)$ and $z(x)$ at the point $x_0 = 1$.
(d) **(5 points)** Compute Taylor's polynomial of order 2 of the functions $y(x)$ and $z(x)$ at the point $x_0 = 1$.

(4) Consider the function

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) **(5 points)** Is the function f continuous at $(0, 0)$?

(b) **(5 points)** Compute $\nabla f(0, 0)$.

(c) **(5 points)** Is the function f differentiable at $(0, 0)$?

(5) **(10 points)** Consider the function $f(u, v) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the functions $u(x, y, z), v(x, y, z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$f(u, v) = u^2 + uv \qquad \text{and} \qquad u(x, y, z) = e^x + y^2 + z, \quad v(x, y, z) = x^2 + e^{y^2} + \ln(z)$$

And consider the composition $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $h(x, y, z) = f(u(x, y, z), v(x, y, z))$. Use the the chain rule to compute

$$\frac{\partial h}{\partial x}(0, 0, 1), \quad \frac{\partial h}{\partial y}(0, 0, 1), \quad \frac{\partial h}{\partial z}(0, 0, 1)$$

(6) Consider the surface given by the equation

$$x^2y - 5xyz + 2yz = 16$$

(a) **(5 points)** Write the equation of the tangent plane to the surface at the point $p = (-1, 2, 1)$.

(b) **(5 points)** Write the parametric equations of the normal line to the surface at the point $p = (-1, 2, 1)$.