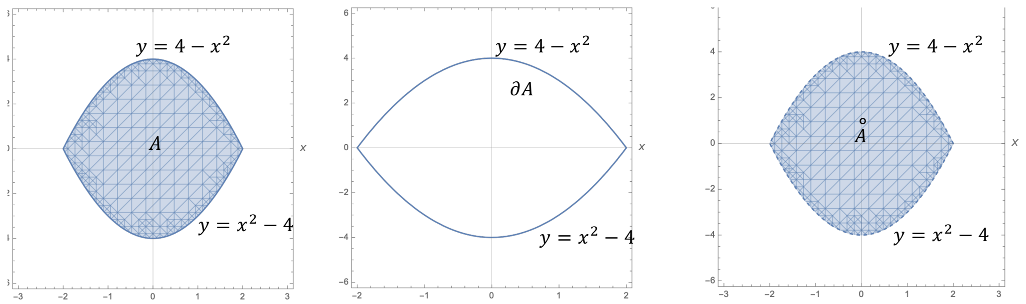
(1) Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 - 4 \le y \le 4 - x^2\}$$

and the function

$$f(x, y) = (2x - y)^3$$

(a) **(10 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.
Solution: *The set A, its interior and its boundary are approximately as indicated in the picture.*



And the closure coincides with A, $\bar{A} = A$. The functions $h(x, y) = x^2 - 4 - y$ and $g(x, y) = x^2 - 4 + y$ are continuous and $A = \{(x, y) \in \mathbb{R}^2 : h(x, y) \le 0, g(x, y) \le 0\}$. Hence, the set A is closed (Note also that $\partial A \subset A$). It is not open because $A \cap \partial A \neq \emptyset$. The set A is bounded. Therefore, the set A is compact. It is also convex .

(b) **(5 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f(x, y) = (2x - y)^3$ defined on A.
Solution: *The set A is compact and the function $f(x, y) = (2x - y)^3$ is continuous. Hence, Weierstrass Theorem applies.*

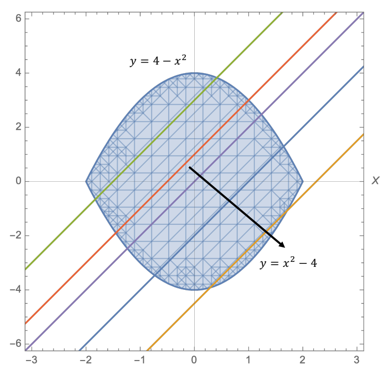
(c) **(5 points)** Draw the level curves of f, indicating the direction of growth of the function.
Solution: *The level curves*

$$f(x, y) = (2x - y)^3 = D$$

are straight lines of the form

$$y = 2x - D^{1/3}$$

Graphically,



The black arrow represents the direction of growth of the function f.

(d) **(10 points)** Using the level curves of f, determine (if it exists) the **global minimum** of f on the set A.

Solution: *Graphically the minimum value is attained at the point* $(x_0, f(x_0))$ *where the line* $y = 2x - D^{1/3}$ *is tangent to the graph of the function* $g(x) = 4 - x^2$. *The slope of the line* $y = 2x - D^{1/3}$ *is* $m = 2$. *Thus* $g'(x_0) = -2x_0 = -2$. *Therefore* $x_0 = -1$, $y_0 = 4 - x_0^2 = 3$. *The minimum value is attained at the point* $(-1, 3)$ *and the minimum value of the function is* $f(-1, 3) = (-5)^3 = -125$.

(2) Consider the function $f(x, y, z) = 2abyz + ax^2 + 2axy + 2ay^2 + cz^2 + 3x + y + 15z - 73$ defined in $\mathbb{R}^3$, with $a, b, c \in \mathbb{R}$ $abc \neq 0$.

(a) **(8 points)** Determine for which values of a, b, c the function f is strictly convex. Determine for which values of a, b, c the function f is strictly concave.

Solution: *We have*

$$\nabla f(x, y, z) = (2ax + 2ay + 3, 2abz + 2ax + 4ay + 1, 2aby + 2cz + 15), \quad \mathrm{H}(f)(x, y, z) = \begin{pmatrix} 2a & 2a & 0 \\ 2a & 4a & 2ab \\ 0 & 2ab & 2c \end{pmatrix}$$

We obtain $D_1 = 2a$, $D_2 = 4a^2 > 0$, $D_3 = |A| = 8a^2c - 8a^3b^2 = 8a^2\left(c - ab^2\right)$. *We see that for* $a > 0$ *and* $c > ab^2$ *we have that* $D_1, D_2, D_3 > 0$ *and the function is convex. For* $a < 0$ *and* $c < ab^2$ *we have that* $D_1 < 0, D_2 > 0, D_3 < 0$ *and the function is concave.*

(b) **(2 points)** Using the results above, determine if the set $D = \{(x, y, z) \in \mathbb{R}^3 : -x^2 - 2xy + 3x - 2y^2 - 4yz + y - 5z^2 + 15z \geq 0\}$ is convex.

Solution:

Taking $a = -1$, $b = 2$, $c = -5$ *we obtain the function* $f(x, y, z) = -x^2 - 2xy + 3x - 2y^2 - 4yz + y - 5z^2 + 15z - 73$. *Since* $c < ab^2$, *the function* f *is concave and the set* D *is convex.*

(3) Consider the system of equations

$$\begin{aligned} 2xy + xz^2 &= 1 \\ xy^2 + z &= -1 \end{aligned}$$

(a) **(5 points)** Using the implicit function theorem, prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.

Solution: *We first remark that* $(x_0, y_0, z_0) = (1, 0, -1)$ *is a solution of the system of equations. The functions* $f_1(x, y, z) = 2xy + xz^2$ *and* $f_2(x, y, z) = xy^2 + z$ *are of class* C^∞. *We compute*

$$\left| \frac{\partial(f_1, f_2)}{\partial(y, z)} \right|_{(x,y,z)=(1,0,-1)} = \begin{vmatrix} 2x & 2xz \\ 2xy & 1 \end{vmatrix}_{(x,y,z)=(1,0,-1)} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} = 2$$

By the implicit function theorem, the above system of equations determines implicitly two differentiable functions $y(x)$ *and* $z(x)$ *in a neighborhood of the point* $(x_0, y_0, z_0) = (1, 0, -1)$.

(b) **(10 points)** Compute

$$y'(1), z'(1)$$

Solution: *Differentiating implicitly with respect to* x,

$$\begin{aligned} 0 &= 2xy'(x) + 2y(x) + 2xz(x)z'(x) + z(x)^2 \\ 0 &= 2xy(x)y'(x) + y(x)^2 + z'(x) \end{aligned}$$

We plug in the values $(x_0, y_0, z_0) = (1, 0, -1)$ *to obtain*

$$\begin{aligned} 0 &= 2y'(1) - 2z'(1) + 1 \\ 0 &= z'(1) \end{aligned}$$

Therefore

$$y'(1) = -\frac{1}{2}, \quad z'(1) = 0$$

(c) **(5 points)** Compute Taylor's polynomial of order 1 of the functions $y(x)$ and $z(x)$ at the point $x_0 = 1$.

Solution: *Taylor's polynomial of order 1 of the function* $y(x)$ *at the point* $x_0 = 1$ *is*

$$P_1(x) = y(x_0) + y'(x_0)(x - x_0) = \frac{1 - x}{2}$$

Taylor's polynomial of order 1 of the function $z(x)$ *at the point* $x_0 = 1$ *is*

$$P_1(x) = z(x_0) + z'(x_0)(x - x_0) = -1$$

(4) Consider the function $f(x,y) = -ay + xy^3 - 2xy + 4x - y^2 + 1$, the point $p = (-1,1)$ and the vector $v = (5,3)$. Here $a \in \mathbb{R}$.

(a) **(5 points)** Compute the gradient of f at the point p. Compute the vector $u = (u_0, u_1)$ with $u_0^2 + u_1^2 = 1$ such that $D_u f(p)$ attains the largest value. Compute the vector $w = (w_0, w_1)$ with $w_0^2 + w_1^2 = 1$ such that $D_w f(p)$ attains the least value.

Solution: *We have*

$$\nabla f(x,y) = \big(y^3 - 2y + 4, -a + 3xy^2 - 2x - 2y\big)$$

Hence,

$$\nabla f(-1,1) = (3, -a-3)$$

And

$$u = \frac{1}{\sqrt{9+(a+3)^2}}(3,-a-3), \quad w = \frac{1}{\sqrt{9+(a+3)^2}}(-3,a+3)$$

(b) **(5 points)** Compute $D_v f(p)$. Compute the value of a if we now that v is perpendicular to the tangent plane to f at the point p.

Solution: *Since,*

$$D_v f(p) = v \cdot \nabla f(p) = (3, -a-3) \cdot (5,3) = 6 - 3a = 0$$

we have that $a = 2$.

(c) **(5 points)** Assuming that $a = 2$, compute the equation of the tangent plane to the graph of the function f at the point $(p, f(p))$.

Solution: *The equation of the tangent plane is*

$$\begin{aligned} z &= f(-1,1) + \nabla f(p) \cdot (x+1, y-1) = -5 + (3,-5) \cdot (x+1, y-1) = \\ &= -5 + 3(x+1) - 5(y-1) \end{aligned}$$

(d) **(5 points)** Assuming that $a = 2$, compute the Hessian matrix of the función f at the point p. Compute Taylor's polynomial of second order of the function f at the point p.

Solution: *The hessian matrix is*

$$\mathrm{H}\, f(x,y) = \begin{pmatrix} 0 & 3y^2 - 2 \\ 3y^2 - 2 & 6xy - 2 \end{pmatrix}$$

Hence,

$$\mathrm{H}\, f(-1,1) = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}$$

Taylor's second order polynomial of the function f at the point p is

$$\begin{aligned} P_2(x,y) &= f(-1,1) + \nabla f(p) \cdot (x+1, y-1) + \frac{1}{2}(x+1, y-1)\, \mathrm{H}\, f(-1,1) \begin{pmatrix} x+1 \\ y-1 \end{pmatrix} = \\ &= -5 + 3(x+1) - 5(y-1) + \frac{1}{2}\big(0 \cdot (x+1)^2 + 2(x+1)(y-1) - 8(y-1)^2\big) \\ &= xy + 2x - 4y^2 + 4y - 2 \end{aligned}$$

(5) **(5 points)** Consider the function $f(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and the functions $x(u,v), y(u,v), z(u,v) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y,z) = x^2y + xz \qquad \text{and} \qquad x(u,v) = e^u, \quad y(u,v) = uv, \quad z(u,v) = \ln v$$

And consider the composition $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $h(u,v) = f(x(u,v), y(u,v), z(u,v))$. Use the the chain rule to compute

$$\frac{\partial h}{\partial u}(0,1), \quad \frac{\partial h}{\partial v}(0,1)$$

at the point $(u_0, v_0) = (0,1)$.

Solution:

$$x(0,1) = 1, \quad y(0,1) = 0, \quad z(0,1) = 0$$

$$Df(x,y,z) = \left(2xy + z, x^2, x\right), \quad Df(1,0,0) = (0,1,1)$$

Let $g(u,v) = (x(u,v), y(u,v), z(u,v)) = (uv, u - v, u + 2v)$. *We have*

$$Dg(u,v) = \begin{pmatrix} e^u & 0 \\ v & u \\ 0 & \frac{1}{v} \end{pmatrix}, \quad Dg(0,1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

By the chain rule,

$$\begin{pmatrix} \frac{\partial h}{\partial u}(0,1) & \frac{\partial h}{\partial v}(0,1) \end{pmatrix} = Df(1,0,0)\, Dg(0,1) = (0,1,1) \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = (1,1)$$

Therefore,

$$\frac{\partial h}{\partial u}(0,1) = 1 \quad \frac{\partial h}{\partial v}(0,1) = 1$$

(6) Consider the surface given by the equation

$$3x^2 + 2y^2 + 5z^2 = 56$$

(a) **(5 points)** Write the equation of the tangent plane to the surface at the point $p = (-1, 2, -3)$.

Solution:
The equation of the normal line is

$$x = -1 - 6t, \quad y = 2 + 8t, \quad z = -3 - 30t$$

(b) **(5 points)** Write the parametric equations of the normal line to the surface at the point $p = (-1, 2, -3)$.

Solution:
The equation of the normal line is

$$x = -1 - 6t, \quad y = 2 + 8t, \quad z = -3 - 30t$$