University Carlos III
Mathematics for Economics I.
Final Exam. January 20th 2023

Last Name:	**Name:**
Degree:	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
6	
Total	

(1) Consider the set

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 - 4 \leq y \leq 4 - x^2\}$$

and the function

$$f(x, y) = (2x - y)^3$$

(a) **(10 points)** Sketch the graph of the set A, its boundary and its interior and justify if it is open, closed, bounded, compact or convex.
(b) **(5 points)** State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function $f(x, y) = (2x - y)^3$ defined on A.
(c) **(5 points)** Draw the level curves of f, indicating the direction of growth of the function.
(d) **(10 points)** Using the level curves of f, determine (if it exists) the **global minimum** of f on the set A.

(2) Consider the function $f(x, y, z) = 2abyz + ax^2 + 2axy + 2ay^2 + cz^2 + 3x + y + 15z - 73$ defined in $\mathbb{R}^3$, with $a, b, c \in \mathbb{R}$ $abc \neq 0$.
(a) **(8 points)** Determine for which values of a, b, c the function f is strictly convex. Determine for which values of a, b, c the function f is strictly concave.

(b) **(2 points)** Using the results above, determine if the set $D = \{(x, y, z) \in \mathbb{R}^3 : -x^2 - 2xy + 3x - 2y^2 - 4yz + y - 5z^2 + 15z \geq 0\}$ is convex.

(3) Consider the system of equations

$$\begin{aligned} 2xy + xz^2 &= 1 \\ xy^2 + z &= -1 \end{aligned}$$

(a) **(5 points)** Using the implicit function theorem, prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x_0, y_0, z_0) = (1, 0, -1)$.

(b) **(10 points)** Compute

$$y'(1), z'(1)$$

(c) **(5 points)** Compute Taylor's polynomial of order 1 of the functions $y(x)$ and $z(x)$ at the point $x_0 = 1$.

(4) Consider the function $f(x, y) = -ay + xy^3 - 2xy + 4x - y^2 + 1$, the point $p = (-1, 1)$ and the vector $v = (5, 3)$. Here $a \in \mathbb{R}$.

(a) **(5 points)** Compute the gradient of f at the point p. Compute the vector $u = (u_0, u_1)$ with $u_0^2 + u_1^2 = 1$ such that $D_u f(p)$ attains the largest value. Compute the vector $w = (w_0, w_1)$ with $w_0^2 + w_1^2 = 1$ such that $D_w f(p)$ attains the least value.

(b) **(5 points)** Compute $D_v f(p)$. Compute the value of a if we now that v is perpendicular to the level curve of f that goes though the point p. That is, v is perpendicular to the curve $\{(x, y) \in \mathbb{R}^2 : f(x, y) = f(p)\}$.

(c) **(5 points)** Assuming that $a = 2$, compute the equation of the tangent plane to the graph of the function f at the point $(p, f(p))$.

(d) **(5 points)** Assuming that $a = 2$, compute the Hessian matrix of the función f at the point p. Compute Taylor's polynomial of second order of the function f at the point p.

(5) **(10 points)** Consider the function $f(x, y, z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and the functions $x(u, v), y(u, v), z(u, v) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y, z) = x^2y + xz \qquad \text{and} \qquad x(u, v) = e^u, \quad y(u, v) = uv, \quad z(u, v) = \ln v$$

And consider the composition $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $h(u, v) = f(x(u, v), y(u, v), z(u, v))$. Use the the chain rule to compute

$$\frac{\partial h}{\partial u}(0, 1), \quad \frac{\partial h}{\partial v}(0, 1)$$

(6) Consider the surface given by the equation

$$3x^2 + 2y^2 + 5z^2 = 56$$

(a) **(5 points)** Write the equation of the tangent plane to the surface at the point $p = (-1, 2, -3)$.

(b) **(5 points)** Write the parametric equations of the normal line to the surface at the point $p = (-1, 2, -3)$.