Consider the function \( f(x) = x \ln x \).

(a) Draw the graph of the function, obtaining firstly its domain, the intervals where \( f(x) \) increases and decreases, its asymptotes and image.

(b) Consider the function \( f(x) \) defined just in the interval \([1/e, \infty)\). Draw the graph of the function \( f^{-1}(x) \), obtaining firstly its domain, image, the intervals where \( f^{-1}(x) \) increases and decreases, the x-intercepts and y-intercepts and its fixed points.

Hint 1: Don’t find the analytical expression of \( f^{-1}(x) \).

Hint 2: The fixed points of \( f(x) \) and \( f^{-1}(x) \) are the same.

a) The domain of \( f \) is the interval \((0, \infty)\). There are no vertical asymptotes, because at \( x = 0 \),
\[
\lim_{x \to 0^+} f(x) = 0 \cdot (\infty) = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \infty = \text{L’Hospital’s Rule} = \]
\[
= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0
\]
There are neither horizontal nor oblique asymptotes at \( \infty \), because \( \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} f(x) = \infty \).
On the other hand, since \( f'(x) = 1 + \ln x \), and \( 1 + \ln x = 0 \iff \ln x = -1 \iff x = e^{-1} = 1/e \), we notice that \( f' \) is increasing on \((0, \infty)\), is negative on \((0, 1/e)\) and positive on \((1/e, \infty)\) then we know that \( f \) is decreasing on the interval \((0, 1/e]\) and increasing on \([1/e, \infty)\).
Finally, since \( f(1/e) = (1/e) \ln(1/e) = -1/e \), \( \lim_{x \to \infty} f(x) = \infty \), \( \lim_{x \to 0^+} f(x) = 0 \) and the function is continuous on \((0, \infty)\) then its range is \([-1/e, \infty)\).
So, we can conclude after all that the graph on the function \( f(x) \) is the first figure:

![Graph of f(x) = x ln x](image1.png)

b) Knowing that the function \( f(x) \), is continuous and increasing on \([1/e, \infty)\) and its range \([-1/e, \infty)\) this means that the inverse function \( f^{-1}(x) \) is increasing in its domain, the interval \([-1/e, \infty)\), and its range will be \([1/e, \infty)\).
Because the only x-intercept and y-intercept point of the restricted function is \((1, 0)\), then \( f(1) = 0 \), and the unique equivalent point of the inverse will be \((0, 1)\), so \( f^{-1}(0) = 1 \).
Finally, there is only one fixed point for both \( f \) and \( f^{-1} \) that is \( x = e \), since \( f(x) = x \iff \ln x = 1 \iff x = e \). So, the graph of the function \( f^{-1}(x) \) will be approximately as you can see in the second figure:

![Graph of f^{-1}(x) = e^{x/e}](image2.png)
(2) Let \( y = f(x) \) be an implicit function defined by the equation \( axy + 2 = 2xe^y + x^2y \), in a neighborhood of the point \((1,0)\) and \( a \neq 3 \).

(a) Depending on the parameter \( a \), calculate the derivative of \( f(x) \) at \((1,0)\).

What are the values of \( a \), such that the function is increasing or decreasing near \( x = 1 \)?

(b) Depending on the parameter \( a \), find the tangent line of \( f \) at \((1,0)\). Give the values of \( a \), when that tangent line is parallel and perpendicular to the angle bisector of the first quadrant \((y = x)\).

1 point

a) Firstly, we compute the first derivative of the equation with respect to \( x \),
\[
ay + axy' = 2e^y + 2xy' + 2xy + x^2y'
\]
Next we substitute \( x = 1, y = 0 \) in order to obtain:
\[
ay' = 2 + 2y' + y' \iff (a - 3)y' = 2 \iff y' = 2/(a - 3).
\]
Consequently, the implicit function will behave in a neighbourhood of \( x = 1 \) increasingly when \( a > 3 \) and decreasingly if \( a < 3 \).

b) The equation of the tangent line is:
\[
y - 0 = (2/(a - 3))(x - 1).
\]
Then, that line will be parallel to \( y = x \) if \( \frac{2}{a - 3} = 1 \iff a = 5 \).
In the same way, that line will be parallel to \( y = x \) if \( \frac{2}{a - 3} = -1 \iff a = 1 \)
(3) Let $C(x) = 2x^2 - 3x + C_0$ be the cost function and $p(x) = 197 - 2x$ the (inverse) demand function of a monopolistic firm, that produces at least two units.

(a) Find the production $x_0$ that maximizes the profit of the firm.

(b) Suppose that $x_1 = 2x_0$ is the production that minimizes the average cost for the company, find $C_0$.

**Remark:** Justify all your answers.

1 point

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a) The profit function is:

$$B(x) = I(x) - C(x) = 197x - 2x^2 - (2x^2 - 3x + C_0) = -4x^2 + 200x - C_0$$

and its critical point $x_0$ is:

$$B'(x_0) = -8x_0 + 200 = 0 \iff x_0 = 25.$$  

Thus, because $B''(x) < 0$ in its domain, the profit function is strictly concave then the critical point is the unique global maximizer point for the profits.

b) Firstly, the average cost function is $C(x) = \frac{C_0}{x} - \frac{3 + 2x}{x}$ and its first order derivative is

$$\left( \frac{C(x)}{x} \right)' = -\frac{C_0}{x^2} + 2$$

Because $x_1 = 2x_0 = 50$ has to be a critical point of the average cost function:

$$-\frac{C_0}{50^2} + 2 = 0 \iff C_0 = 2.50^2 = 5.000.$$  

Then, because $\left( \frac{C(x)}{x} \right)'' > 0$ the average cost function is strictly convex in its domain and the critical point will be the unique global minimizer for the function.
Let $a, b$ be two real numbers, and consider the following piecewise function

$$f(x) = \begin{cases} 
  e^{a+x} & \text{if } x < -3 \\
  1 & \text{if } x = -3 \\
  \sqrt{x+b} & \text{if } x > -3 
\end{cases}$$

(a) Discuss the continuity of the function $f$ with respect to the numbers $a, b$.

(b) Discuss the differentiability of the function $f$ with respect to the numbers $a, b$.

1 point

(a) For any value of $a$, the function is continuous on $x < -3$.

Furthermore, at $x = -3$, the function is continuous from the left if:

$$\lim_{x \to -3^-} f(x) = f(-3) \iff e^{a-3} = 1 \iff a = 3.$$  

The function is continuous on $x > -3$ if $b \geq -3$.

And in particular, $f$ is continuous at $x = -3$ from the right if:

$$\lim_{x \to -3^+} f(x) = f(-3) \iff \sqrt{b-3} = 1 \iff b = 4.$$  

Consequently, $f(x)$ is continuous for every $x$ when $a = 3, b = 4$.

Finally, we can notice that for those values $a = 3, b = 4$, the function is continuous for every point in its domain.

(b) Obviously, when $x \neq -3$ our function is differentiable in its domain, because the function is an exponential or a square root function and both are differentiable.

At the point $x = -3$, we need to calculate the one-sided derivatives, knowing that the function is continuous at the point when $a = 3, b = 4$.

$$f'_-(3) = \lim_{x \to -3^-} f'(x) = \lim_{x \to -3^-} e^{x+3} = e^0 = 1$$

$$f'_+(3) = \lim_{x \to -3^+} f'(x) = \lim_{x \to -3^+} \frac{1}{2\sqrt{x+4}} = \frac{1}{2}.$$  

Then the function will never be differentiable at $x = -3$. 
(5) Consider the set of points in the plane $A$ bounded by the curves $y = \ln(x + 4), y = -e^x$ and the straight lines $x = 0$, $x = 2$.

(a) Draw the set $A$ and find, if they exist, the maximals, minimals, maximum and minimum points of $A$.

(b) Calculate the area of the region $A$.

Hint 1: Pareto order is defined by: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1$ and $y_0 \leq y_1$.

Hint 2: Do not calculate the values either of the logarithms (ln) or of the exponentials (exp) in the solutions.

1 point

a) The function $f(x) = \ln(x + 4)$ is positive and increasing on the interval $[0, 2]$, and the function $g(x) = -e^x$ is negative and decreasing on the real line, so the set $A$ has a shape which is approximately like this:

![Diagram of the set A](image)

Clearly, we can deduce looking at the drawing that

$\{\text{maximals of } (A)\} = \{\text{maximum of } (A)\} = \{(2, \ln 6)\}$

$\{\text{minimals of } (A)\} = \{(x, y) : 0 \leq x \leq 2, y = -e^x\}$; the minimum of $(A)$ doesn’t exist.

b) The required area is below the logarithm function and above the exponential one and inbetween the vertical lines $x = 0, x = 2$.

So, the area is: $\int_0^2 (\ln(x + 4) + e^x) dx$

Integrating by parts:

$\int \ln(x + 4) dx = x \ln(x + 4) - \int \frac{x}{x + 4} dx = x \ln(x + 4) - \int \frac{x}{x + 4} dx$

$= x \ln(x + 4) - x + 4 \ln(x + 4) = (x + 4) \ln(x + 4) - x.$

And applying Barrow’s Rule we obtain:

$\left[ (x + 4) \ln(x + 4) - x + e^x \right]_0^2 = 6 \ln 6 - 2 + e^2 - (4 \ln 4 + 1)$

$= 6 \ln 6 + e^2 - 4 \ln 4 - 3$ area units.
Given the function $f(x) = \frac{x}{\sqrt{x} - 1}$ defined in the interval $(1, \infty)$.

(a) Find the primitive function $F(x)$ of $f$ such that satisfies $F(2) = \frac{8}{3}$.

(b) Prove the inequality $\sqrt{x + 1} < \frac{x}{\sqrt{x} - 1}$ and use it to find a lower bound of the definite integral $\int_3^8 \frac{x}{\sqrt{x} - 1} dx$.

1 point

a) Making the change of variable $x - 1 = t^2$, $dx = 2tdt$, we can deduce that the primitive function of $f$ will be:

$$F(x) = \int \frac{x}{\sqrt{x} - 1} dx = \int \frac{1 + t^2}{t} 2tdt = 2 \int (1 + t^2) dt = 2(t + t^3/3) + C =$$

$$= 2\sqrt{x - 1} + \frac{2}{3} \sqrt{(x - 1)^3} + C$$

Because $\frac{8}{3} = F(2) = \frac{8}{3} + C$, we know that $C = 0$.

So, the answer is $F(x) = 2\sqrt{x - 1} + \frac{2}{3} \sqrt{(x - 1)^3}$.

b) Multiplying the inequality by the denominator we get the equivalent inequation:

$$\sqrt{x + 1} \sqrt{x - 1} < x$$

squaring both sides we obtain:

$$(x + 1)(x - 1) = x^2 - 1 < x^2$$

so, we can see that the inequality is satisfied. From that fact we can deduce that $\int_3^8 \sqrt{x - 1} dx < \int_3^8 \frac{x}{\sqrt{x} - 1} dx$.

Thus, because the value of the first integral is:

$$\int_3^8 \sqrt{x - 1} dx = \left[ \frac{2}{3} (x + 1)^{3/2} \right]_3^8 = \frac{2}{3}(27 - 8) = \frac{38}{3}$$

we can state that $\frac{38}{3} < \int_3^8 \frac{x}{\sqrt{x} - 1} dx$. 