1. Consider the function \( f(x) = \frac{\ln(1 + 2x)}{1 + 2x} \). You are asked to

(a) Draw the graph of the function, obtaining firstly its domain, the intervals where \( f \) increases and decreases, its asymptotes, and image.

(b) Consider the functions \( f_1(x) := f(x) \) defined just in the interval where \( f \) is increasing, and \( f_2(x) := f(x) \) defined just in the interval where \( f \) is decreasing. Find the domains and the images of the functions \( f_1^{-1} \) and \( f_2^{-1} \), and draw their graphs.

Suggestion: Do not try to compute the analytical expressions of \( f_1^{-1} \) and \( f_2^{-1} \).

(a) The domain of \( f \) is \( D = \{ x \in \mathbb{R} : 1 + 2x > 0 \} = ( -\frac{1}{2}, \infty ) \). The first derivative is

\[
f'(x) = \frac{\frac{2}{1+2x}(1 + 2x) - 2 \ln(1 + 2x)}{(1 + 2x)^2} = \frac{2(1 - \ln(1 + 2x))}{(1 + 2x)^2}
\]

One has \( f'(x) = 0 \) holds iff \( \ln(1 + 2x) = 1 \), that is, \( 1 + 2x = e \). So, the unique critical point is \( x^* = e - \frac{1}{2} \).

\[
f(x^*) = \frac{\ln(1 + 2e^*-1)}{1 + 2e^*-1} = \frac{\ln(1 + e - 1)}{1 + e - 1} = \frac{1}{e}
\]

- \( f'(x) \geq 0 \iff 1 \geq \ln(1 + 2x) \iff e \geq 1 + 2x \iff x^* \geq x \). So, \( f \) is increasing in \( ( -\frac{1}{2}, e - \frac{1}{2} ] \)
- \( f'(x) \leq 0 \iff 1 \leq \ln(1 + 2x) \iff e \leq 1 + 2x \iff x^* \leq x \). So, \( f \) is decreasing in \( [ e - \frac{1}{2}, \infty ) \)

Consequently, \( f \) has a local maximum in the point \((x^* , f(x^*)) = ( e - \frac{1}{2}, \frac{1}{e} ) \). In fact, it is global.

Since \( f \) is continuous on its domain, we just study the asymptotes at \( -\frac{1}{2}^+ \) and \( +\infty \).

\[
\lim_{x \to -\frac{1}{2}^+} f(x) = -\infty = -\infty; \quad \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\ln(1 + 2x)}{1 + 2x} = \frac{\infty}{\infty} \text{ L'Hôpital } \Rightarrow \lim_{x \to +\infty} \frac{\frac{2}{1+2x}}{2} = 0;
\]

Hence, \( f \) has a vertical asymptote in \( x = \frac{1}{2} \) (on the right) and an horizontal asymptote in \( y = 0 \) (in \( +\infty \)).

There are no oblique asymptotes.

Then, the image of the function is \( ( -\infty, \frac{1}{e} ] \) and its graph is as follows
(b) By definition $f_1(x) = f(x)$ for all $x \in (-\frac{1}{2}, \frac{e-1}{2})$ and it is a bijective and increasing function. Hence,

$$f_1 : \left(-\frac{1}{2}, \frac{e-1}{2}\right] \rightarrow \left(-\infty, \frac{1}{e}\right] \quad \text{and so} \quad f_1^{-1} : \left(-\infty, \frac{1}{e}\right] \rightarrow \left(-\frac{1}{2}, \frac{e-1}{2}\right]$$

The function $f_1^{-1}$ is also bijective and increasing, and its graph is

![Graph of $f_1$ and $f_1^{-1}$]

By definition $f_2(x) = f(x)$ for all $x \in [\frac{e-1}{2}, +\infty)$ and it is a bijective and decreasing function. Hence,

$$f_2 : \left[\frac{e-1}{2}, +\infty\right) \rightarrow \left(0, \frac{1}{e}\right] \quad \text{and so} \quad f_2^{-1} : \left(0, \frac{1}{e}\right] \rightarrow \left[\frac{e-1}{2}, +\infty\right)$$

The function $f_2^{-1}$ is also bijective and decreasing, and its graph is

![Graph of $f_2$ and $f_2^{-1}$]
2. Given the function \( y = f(x) \) implicitly defined by the equation \( y + e^x + y = 0 \) in a neighborhood of the point \( x = 1, y = -1 \), you are asked to (10 points)

(a) Find the second-order Taylor polynomial of \( f(x) \) around \( a = 1 \). Use it to get an approximation of \( f(0, 9) \).

(b) Find the equation of the tangent line to \( f \) at the point \( x = 1 \). Draw a sketch of the graph of \( f \) around the point \( x = 1 \).

Hint: Use the fact that \( f'(1) < 0 \) and \( f''(1) < 0 \).

(a) Firstly, we compute the first and second derivatives of the function

\[
\begin{align*}
\frac{d}{dx}(f(x) + e^x + f(x)) &= 0 \\
(1 + f'(x))e^x + f'(x)e^x &= 0 \\
(1 + f'(x))^2e^x + f''(x)e^x &= 0
\end{align*}
\]

Next we substitute \( x = 1 \), \( f(1) = -1 \),

\[
\begin{align*}
f'(1) + (1 + f'(1))e^0 &= 0 \\
f''(1) + f''(1)e^0 + (1 + f'(1))^2e^0 &= 0
\end{align*}
\]

Consequently, \( f'(1) = -\frac{1}{2} \) and \( f''(1) = -\frac{1}{8} \).

So, the second-order Taylor polynomial around \( a = 1 \) is

\[
P(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 = -1 - \frac{1}{2}(x - 1) - \frac{1}{16}(x - 1)^2
\]

\[
f(0, 9) \approx P(0, 9) = -1 + \frac{1}{2}(0, 1) - \frac{1}{16}(0, 1)^2 = \frac{-1600 + 80 - 1}{1600} = \frac{-1521}{1600}.
\]

(b) Finally, the equation of the tangent line to \( f \) at the point \( x = 1 \) is

\[
y = -1 - \frac{1}{2}(x - 1) = \frac{-x - 1}{2}
\]

Since \( f''(1) < 0 \), the function \( f \) is concave in a neighborhood of \( x = 1 \). Hence, the graph of \( f \) lies below the given tangent line around the point \( x = 1 \).
3. Let \( C(x) = C_0 + 40x + 0.04x^2 \) be the cost function of a monopolistic firm, where \( C_0 > 0 \) represents the fixed costs and \( x \geq 0 \) is the output. The inverse demand function is given by \( p(x) = 60 - 0.06x \). You are asked to (10 points)

(a) Find the price \( p^* \) that maximizes the benefit of the firm. Justify why it gives the maximum benefit.

(b) Find the value of \( C_0 \) such that the level of output that maximizes the benefit coincides with the level of output that minimizes the average costs. In such a case, which is the benefit? And the average cost?

(a) The benefit function is

\[
B(x) = P(x)x - C(x) = 60x - 0.06x^2 - (C_0 + 40x + 0.04x^2) = -0.1x^2 + 20x - C_0.
\]

One has

\[
B'(x) = -0.2x + 20 \quad \text{and} \quad B''(x) = -0.2 < 0
\]

\( B \) is a concave function and it has a unique critical point in \( x^* = 100 \), so that point is a global maximizer.

The price associated to this level of output is \( p^* = p(x^*) = p(100) = 60 - 0.06 \cdot 100 = 54 \).

(b) The average cost function is

\[
CM(x) = \frac{C(x)}{x} = \frac{C_0}{x} + 40 + 0.04x
\]

One has

\[
CM'(x) = -\frac{C_0}{x^2} + 0.04 \quad \text{and} \quad CM''(x) = \frac{2C_0}{x^3} > 0
\]

\( CM \) is a convex function and it has a unique critical point in \( \tilde{x} = \sqrt[3]{\frac{C_0}{0.04}} \), so that point is a global minimizer.

By hypothesis, the level of output that maximizes the benefit coincides with the level of output that minimizes the average costs, that is, \( x^* = \tilde{x} \) and so

\[
\sqrt[3]{\frac{C_0}{0.04}} = 100 \quad \Rightarrow \quad C_0 = 0.04 \cdot 100^2 = 400
\]

For that value of \( C_0 \), the maximum benefit of the firm is

\[
B(x^*) = B(100) = -0.1 \cdot 100^2 + 20 \cdot 100 - 400 = 600,
\]

whereas the minimum average cost is

\[
CM(\tilde{x}) = 4 + 40 + 4 = 48
\]
4. Let \( f : [0, 3] \to \mathbb{R} \) be a continuous function in \([0, 3]\) and twice derivable in \((0, 3)\) such that
\[
f(0) = 1, \quad f(1) = 2, \quad f(2) = 4, \quad f(3) = 8.
\]
You are asked to (10 points)
(a) Prove that there exist \( c_1 \in (0, 1) \) such that \( f'(c_1) = 1 \) and \( c_2 \in (2, 3) \) such that \( f''(c_2) = 4 \).
(b) Prove that there exists \( c_3 \in (0, 3) \) such that \( 1 < f''(c_3) < 3 \).

Hint: Apply the Lagrange’s Theorem to the appropriate function in the appropriate interval, and find a lower bound and an upper bound for \( c_2 - c_1 \).

\[\text{(a)}\] By applying the Lagrange Theorem to \( f \) in \([0, 1]\), we get the existence of \( c_1 \in (0, 1) \) such that
\[f'(c_1) = \frac{f(1) - f(0)}{1 - 0} = 1.
\]
Analogously, by applying the Lagrange Theorem to \( f \) in \([2, 3]\), we get the existence of \( c_2 \in (2, 3) \) such that
\[f'(c_2) = \frac{f(3) - f(2)}{3 - 2} = 4.
\]
\[\text{(b)}\] By applying the Lagrange Theorem to \( f' \) in \([c_1, c_2] \subset [0, 3]\), we get the existence of \( c_3 \in (c_1, c_2) \subset (0, 3) \) such that
\[f''(c_3) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{3}{c_2 - c_1}.
\]
Since \( c_1 \in (0, 1) \) and \( c_2 \in (2, 3) \), one has \(-c_1 \in (-1, 0)\) and so \( 1 < c_2 - c_1 < 3 \). Hence,
\[1 > \frac{1}{c_2 - c_1} > \frac{1}{3} \quad \text{and so} \quad 3 > \frac{3}{c_2 - c_1} > 1.
\]
Consequently, \( 1 < f''(c_3) < 3 \).
5. Consider the set \( A = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq f(x)\} \) where \( f \) is an increasing function and convex in the interval \([2, 4]\) and it holds \( f(2) = 5, f'(2) = 3, f(4) = 12\). You are asked to \( \) (10 points)

(a) Draw the set \( A \) and find, if they exist, the maximal, minimal, maximum and minimum points of \( A \). Recall that the Pareto order is defined by \((x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1\).

(b) Find the best approximations (one from below and the other from above) of the area of the set \( A \).

Hint: Draw the graph of the function, the tangent line to \( f \) in \((2, f(2))\), and the straight line that crosses points \((2, f(2))\) and \((4, f(4))\).

Remark: the difference between both approximations is 1 unit area.

(a) Graph is shown in part (b). Since the function \( f \) is positive and increasing in \([2, 4]\), one has

\[
\text{maximum}(A) = \text{maximals}(A) = \{(4, f(4))\}
\]

\[
\text{minimum}(A) = \text{minimals}(A) = \{(2, 0)\}
\]

(b) Due to the convexity, the graph of the function lies above the tangent line to \( f \) in \((2, f(2))\), which is

\[
y - 5 = 3(x - 2) \Rightarrow y = 3x - 1
\]

On the other hand, also due to the convexity, the graph of the function lies below the straight line that crosses points \((2, f(2))\) and \((4, f(4))\), which is \( y = 3.5x - 2 \).

Hence, since \( f \) is positive and increasing, if

\[
F := \text{area}(A) = \int_2^4 f(x)\,dx,
\]

one has

\[
F \geq F_b := \int_2^4 (3x - 1)\,dx = 3\frac{x^2}{2} - x|_2^4 = (24 - 4) - (6 - 2) = 16
\]

\[
F \leq F_a := \int_2^4 (3.5x - 2)\,dx = 3.5\frac{x^2}{2} - 2x|_2^4 = (28 - 8) - (7 - 4) = 17
\]

Another way to get these estimations is the following one. Observe that \( F_b \) is the area of the rectangle with width 2 (the length of the interval \([2, 4]\)) and height 5 (\( f(2) \)) plus the area of the right triangle which is above the previous rectangle and has the same width, 2, and height 6 (the distance from \((4, 5)\) to \((4, 11)\)). Hence,

\[
F_b = 2 \cdot 5 + \frac{2 \cdot 6}{2} = 10 + 6 = 16
\]

On the other hand, \( F_a \) is the area of the rectangle with width 2 (the length of the interval \([2, 4]\)) and height 5 (\( f(2) \)) plus the area of the right triangle which is above the previous rectangle and has the same width, 2, and height 7 (the distance from \((4, 5)\) to \((4, 12)\)). Hence,

\[
F_a = 2 \cdot 5 + \frac{2 \cdot 7}{2} = 10 + 7 = 17
\]
6. Consider the function 

\[ F(x) = \int_{3}^{x} \frac{2t - 7}{t^2 - t - 2} \, dt \]

defined in \([3, +\infty)\). You are asked to

(a) Find the local extreme points of \(F\) and classify them.
(b) Find the value of \(F(4)\) and justify whether it is positive or negative.

Remark: Statements (a) and (b) are independent each other.

(a) By applying the Fundamental Theorem of Integral Calculus, one has 

\[ F'(x) = \frac{2x - 7}{x^2 - x - 2}. \]

Hence, \(F'(x) = 0\) if and only if \(2x - 7 = 0\) and so, \(x^* = \frac{7}{2}\) is the unique critical point.

Observe that the points \(-1\) and \(2\) where \(F'\) is not defined and so \(F\) would not be differentiable at those points, are not critical points since we are assuming that \(F\) is just defined at \([3, +\infty)\).

Now, we study the second derivative of \(F\) at \(x^*\) to classify that point.

\[ F''(x) = \frac{2(x^2 - x - 2) - (2x - 7)(2x - 1)}{(x^2 - x - 2)^2} = \frac{-2x^2 + 14x - 11}{(x^2 - x - 2)^2} \]

\[ F''(x^*) = \frac{-2(49/4) + 14(7/2) - 11}{((49/4) - (7/2) - 2)^2} = \frac{(-49 + 98 - 22)/2}{(49 - 14 - 22)/4}^2 = \frac{27/2}{27/4} = \frac{8}{27} > 0 \]

Hence, \(F\) attains a local minimum in \(x^* = \frac{7}{2}\).

(b) Since \(t^2 - t - 2 = (t + 1)(t - 2)\), then

\[ \frac{2t - 7}{t^2 - t - 2} = \frac{A}{t + 1} + \frac{B}{t - 2} \quad \Rightarrow \quad 2t - 7 = A(t - 2) + B(t + 1). \]

If we substitute \(t = 2\) then we get \(-3 = 3B\) and so \(B = -1\). Analogously, if we substitute \(t = -1\) we get \(-9 = -3A\) and so \(A = 3\). Hence,

\[ F(4) = \int_{3}^{4} \frac{2t - 7}{t^2 - t - 2} \, dt = \int_{3}^{4} \frac{3}{t - 2} \, dt + \int_{3}^{4} \frac{-1}{t + 1} \, dt = [3 \ln(t + 1) - \ln(t - 2)]_{3}^{4} = \]

\[ = (3 \ln(5) - \ln(2)) - (3 \ln(4) - \ln(1)) = 3 (\ln(5) - \ln(4)) - \ln(2) = 3 \ln \left( \frac{5}{4} \right) - \ln(2) = \]

\[ = \ln \left( \frac{5}{4} \right)^3 - \ln(2) = \ln \left( \frac{125}{64} \right) - \ln(2) = \ln \left( \frac{125}{128} \right) < 0 \]

since \(\frac{125}{128} < 1\).