1. The function $f$ is defined by $f(x) = \begin{cases} 2 - 4(x - 4)^2 & \text{if } 3 \leq x \leq 4 \\ 2 + 4(x - 4)^2 & \text{if } 4 \leq x \leq 5 \end{cases}$.

a) Find the range of $f$ and find the inverse function $f^{-1}$.

b) Sketch the graph of the inverse function $f^{-1}$. Find its domain and its range.

Hint: sketch the graph of the function $f$ and observe if it is an increasing or decreasing function.

1 point.

a) The function $f$ is continuous and strictly increasing on its domain. Finding the values of $f(3) = -2$ and $f(5) = 6$ we know that the range of the function is $[-2, 6]$.

In order to find the inverse function $f^{-1}$, firstly, we consider $f$ defined on the subdomain $[3, 4]$, whose range is the interval $[-2, 2]$.

$y = 2 - 4(x - 4)^2 \iff (x - 4)^2 = \frac{2 - y}{4} \iff x - 4 = -\sqrt{\frac{2 - y}{4}} \iff x = 4 - \frac{\sqrt{2 - y}}{2};$

So, $f^{-1}(x) = 4 - \frac{\sqrt{2 - x}}{2}$, whenever $x \in [-2, 2]$.

Secondly, we consider $f$ defined on the subdomain $[4, 5]$, whose range is the interval $[2, 6]$.

$y = 2 + 4(x - 4)^2 \iff (x - 4)^2 = \frac{y - 2}{4} \iff x - 4 = \sqrt{\frac{y - 2}{4}} \iff x = 4 + \frac{\sqrt{y - 2}}{2};$

So, $f^{-1}(x) = 4 + \frac{\sqrt{x - 2}}{2}$, whenever $x \in [2, 6]$.

Thus we have: $f^{-1}(x) = \begin{cases} 4 - \frac{\sqrt{2 - x}}{2} & \text{si } -2 \leq x \leq 2 \\ 4 + \frac{\sqrt{x - 2}}{2} & \text{si } 2 \leq x \leq 6 \end{cases}$.

b) To sketch the graph of the inverse function $f^{-1}$, it is useful to bear in mind that as $f$ is strictly increasing it will also be its inverse function.

We also know that for the inverse function $f^{-1}$, the domain is $[-2, 6]$, the range is $[3, 5]$ and $f(2) = 4$.

And furthermore, since $f$ is concave on the interval $[3, 4]$ and convex on the interval $[4, 5]$, thus $f^{-1}$ is convex on $[-2, 2]$ and concave on $[2, 6]$.

Thus, the graph of $f^{-1}$ will approximatly be:
2. The function \( f \) is defined by

\[
 f(x) = \begin{cases} 
 x^3, & \text{if } x < 0 \\
 1 + x^2, & \text{if } x = 0 \\
 a, & \text{if } x > 0 \\
 \ln(x^2 + 1), & \text{if } x > 0 
\end{cases}
\]

where \( a \in \mathbb{R} \), is a real number.

a) Find the value of \( a \) giving your justification, if there is any, such that the function \( f \) is continuous and/or differentiable at \( x = 0 \).

b) Find the vertical, horizontal and oblique asymptotes of the function \( f \) for each value of \( a \).

1 point.

a) Firstly, in order to study whether the function is continuous at the point \( x = 0 \), we calculate the sided limits of \( f(x) \) at \( x = 0 \).

\[
 \lim_{x \to 0^-} f(x) = 0, \quad f(0) = a, \quad \lim_{x \to 0^+} f(x) = \frac{0}{0} = (L'Hospital) = \lim_{x \to 0^+} \frac{2x}{x^2 + 1} = 0; 
\]

In this case, \( f \) is continuous at \( x = 0 \) when \( a = 0 \).

Now, if we suppose that \( f(x) \) is continuous at \( x = 0 \), \( f(x) \) is differentiable at \( x = 0 \) when

\[
 \lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x). \quad \text{But:} 
\]

\[
 \lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} \frac{3x^2(1 + x^2) - x^32x}{(1 + x^2)^2} = 0. 
\]

and, \( \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{2x/(x^2 + 1)}{x^2} = \lim_{x \to 0^+} \frac{2x}{x^2} = 1 \).

Calculating the value of both limits apart:

\[
 \lim_{x \to 0^+} \frac{2x/(x^2 + 1)}{x^2} = \lim_{x \to 0^+} \frac{2}{x^2 + 1} = 2. \quad \text{And the other one,} 
\]

\[
 \lim_{x \to 0^+} \frac{\ln(x^2 + 1)}{x^2} = \lim_{x \to 0^+} \frac{2x/(x^2 + 1)}{2x} = 1. 
\]

We finally obtain \( \lim_{x \to 0^+} f'(x) = 1 \).

Thus, we can assert that \( f(x) \), in no case is differentiable at \( x = 0 \).

b) Obviously, vertical asymptotes do not exist. To find asymptotes at infinity:

\[
 \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^3}{x} = 1, 
\]

\[
 \lim_{x \to \infty} f(x) - x = \lim_{x \to \infty} \frac{x^3}{1 + x^2} - \frac{x(1 + x^2)}{1 + x^2} = \lim_{x \to \infty} \frac{-x}{1 + x^2} = 0 
\]

Then \( f \) has an oblique asymptote \( y = x \) at \( -\infty \).

\[
 \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\ln(x^2 + 1)}{x} = \lim_{x \to \infty} \frac{2x}{x^2 + 1} = 0 
\]

Thus the function has a horizontal asymptote \( y = 0 \) at \( \infty \).
3. Consider the equation $x + e^{4x} = b$.

a) Prove that there is always only one solution of the given equation.

b) Find out, when $b = 0$, the solution of the equation with an absolute error less than 0.25.

1 point.

a) The function $f(x) = x + e^{4x}$ is strictly increasing on $\mathbb{R}$, from its first derivative $f'(x) = 1 + 4e^{4x} > 0$.

Therefore, if there is any solution, there is only one.

In order to prove that there is a solution of the equation we should notice that the function is continuous on its domain and from

$$\lim_{x \to -\infty} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = \infty, \quad \text{(range (f) = } \mathbb{R})$$

we can state that the solution always exits for each $b \in \mathbb{R}$ and it is unique.

b) First, we try with $f(0) = 1 > 0$. So since the function is strictly increasing, we know the root is to the left of (i.e. smaller than) $x = 0$.

Since $f(-1) = -1 + e^{-4} = -1 + \frac{1}{e^4} < -1 + \frac{1}{16} < 0$, using Bolzano’s Theorem the root is in the interval $(-1, 0)$.

Finally, if we try the midpoint of the above interval, since $f(-\frac{1}{2}) = -\frac{1}{2} + \frac{1}{e^{2}} < -\frac{1}{2} + \frac{1}{4} < 0$. then we know for the same reason as above, the root is in the interval $(-\frac{1}{2}, 0)$, and in this case we can take the midpoint $x = -\frac{1}{4}$ as an approximate value of the root with an absolute error of less than 0.25.
4. The function $f$ is defined by $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ 1 - 2x & \text{if } 0 \leq x \end{cases}$, and we consider the function $f: [a, b] \rightarrow \mathbb{R}$, where $a < b$ are real numbers.

a) Write down Weierstrass' Theorem and find the values of $a$ and $b$ such that the hypothesis (or initial condition) is true in the theorem.

b) Find the values of $a$ and $b$ such that the hypothesis is NOT satisfied but the thesis (or conclusion) is.

Hint: Draw the graph of the function.

1 point.

a) The hypothesis (conditions) is satisfied when the function is continuous. Therefore this happens when $0 \leq a$ or when $b < 0$.

b) On the one hand the hypothesis is not satisfied when the function $f$ is discontinuous, this is the case when $a < 0 \leq b$.

On the other hand, for each value of $a$ and $b$ the function always attains its global maximum.

In order to determine the solution, we just need to consider the case that $f$ is discontinuous and attains its global minimum. Since the function is only left handed discontinuous at $x = 0$ and since \[ \lim_{x \to 0^-} f(x) = -1 < f(0), \] and we notice that $f(x) \leq -1 \iff x \geq 1$
then, $f(x)$ is discontinuous and satisfies the thesis when $a < 0 < 1 \leq b$.

Look at the graph of $f$ in order to fully understand the situation.
5. Let \( A = \{(x, y) \in \mathbb{R}^2 : x^2 - 2x + 1 \leq y \leq -x^2 + 2x + 1\} \) be a set of points.

a) Draw the set \( A \) and obtain the maximum, the minimum, the maximal elements and minimal elements of the set \( A \) if they exist.

b) Calculate the area of the region given by the set \( A \).

Hint: The Pareto ordering is: \((x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1\).

1 point.

a) \( f(x) = x^2 - 2x + 1 \) describes a convex parabola whose vertex is the point \((1, 0)\), since \( f'(1) = 0, f(1) = 0, f''(1) > 0 \).

\( g(x) = -x^2 + 2x + 1 \) describes a concave parabola whose vertex is the point \((1, 2)\), since \( g'(1) = 0, g(1) = 2, g''(1) < 0 \).

To find the points \((x, y)\) of intersection on both parabolas we solve the equation:

\[ x^2 - 2x + 1 = -x^2 + 2x + 1 \iff 2x^2 = 4x \iff x = 0, x = 2. \]

to obtain: \((0, 1), (2, 1)\).

Thus the region bound by set \( A \) is:

![Graph of set A](image)

It is certain that there are neither \( \max(A) \) nor \( \min(A) \), since

- maximal points of \((A) = \{(x, y) : y = -x^2 + 2x + 1, 1 \leq x \leq 2\}\)
- minimal points of \((A) = \{(x, y) : y = x^2 - 2x + 1, 0 \leq x \leq 1\}\).

b) The area of the region is:

\[
\int_0^2 [(x^2 - 2x + 1) - (2x^2 - 4x + 1)]dx = \int_0^2 (-2x^2 + 4x)dx = \left[ -\frac{2x^3}{3} + 2x^2 \right]_0^2 = \frac{-16}{3} + 8 = \frac{8}{3} \text{ area units.}
\]
6. Given the function \( F(x) = \int_{-2}^{x} t^3 e^{-t^2} \, dt \), defined at \( x \in [-2, 2] \).

a) Find the intervals in which \( F(x) \) is increasing/decreasing. Find the local and/or global maxima and minima of \( F(x) \).

b) Find the intervals in which \( F(x) \) is convex/concave. Locate any inflection points of \( F(x) \).

Notice: It is neither necessary nor helpful to find the primitive function of \( f(x) = x^3 e^{-x^2} \).

1 point.

a) Referring to the Fundamental Theorem of Calculus we know that \( F'(x) = x^3 e^{-x^2} \). Therefore, the following is satisfied:
\[
F'(x) < 0 \iff x < 0; F'(x) > 0 \iff x > 0.
\]
So we know that \( F(x) \) is strictly decreasing on the interval \((-2, 0)\) and strictly increasing on the interval \((0, 2)\).
Firstly we can conclude that \( F(x) \) attains its global (and local) minimum at \( x = 0 \).
And secondly, to find the maximum of the function \( F(x) \), since \( f(x) = x^3 e^{-x^2} \) is an odd function, then
\[
\int_{-2}^{0} t^3 e^{-t^2} \, dt = -\int_{0}^{2} t^3 e^{-t^2} \, dt \text{ and we know that } F(-2) = 0 = F(2).
\]
We can conclude that \( F(x) \) attains its global maximum at the points \( x = -2, x = 2 \).

b) Since \( F''(x) = f'(x) = x^2(-2x^2 + 3)e^{-x^2} \), then:
\[
F''(x) = 0 \iff x = 0, x = \pm \sqrt{\frac{3}{2}}.
\]
Furthermore, since
\[
F''(x) < 0 \iff x \in (-2, -\sqrt{\frac{3}{2}}) \cup (\sqrt{\frac{3}{2}}, 2); \text{ (these are two different open intervals where the function is concave) and}
\]
\[
F''(x) > 0 \iff x \in (-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}); \text{ (this is an open interval where the function is convex}).
\]
We can thus state that the function has two inflection points at \( x = \pm \sqrt{\frac{3}{2}} \).
A sketch of the graph of \( F \) is: