Chapter 4

Applications of the derivatives

4.1 Higher order derivatives

If the derivative $f'$ of the function $f$ is defined in an interval $(c - \delta, c + \delta)$ around point $c$, then the second derivative of $f$ is the derivative of the function $f'$, and it is denoted $f''$. The third derivative is defined as the derivative of the second derivative and so on. The third derivative is denoted $f'''$ and, more generally, the $n$th order derivative by $f^{(n)}$, and once the $(n - 1)$th derivative is computed, it is given by $f^{(n)}(x) = (f^{(n-1)}(x))'$. We will say that a function is of class $C^n$ if the $n$th order derivative of $f$, $f^{(n)}$, exists in an open interval, and $f^{(n)}$ is continuous.

Example 4.1.1. Given the function $f(x) = 4x^4 - 2x^2 + 1$, $f'(x) = 16x^3 - 4x$, $f''(x) = 48x^2 - 4$, $f'''(x) = 96x$, $f^{(4)}(x) = 96$ and $f^{(n)}(x) = 0$ for every $n \geq 5$.

4.2 Taylor polynomial

4.2.1 Taylor polynomial or order 2

Remark: the tangent line or Taylor polynomial of order 1:

$$ y = P_{1,a}(x) = f(a) + f'(a).(x - a) $$

is characterized by the fact that satisfies:

$$ \lim_{x \to a} \frac{f(x) - P_{1,a}(x)}{(x-a)} = 0 $$

what can be proven using L'Hopital rule.

From the limit above we can define the Taylor polynomial.

Definition 4.2.1. The Taylor polynomial of order $n$ is characterized as the unique polynomial of degree $\leq n$ that satisfies: $\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = 0$.

From the limit above can be deduced that, when $n = 2$ :

Theorem 4.2.2. $P_{2,a}(x) = f(a) + f'(a).(x - a) + \frac{f''(a)}{2}(x - a)^2$
Proof: Use L’Hopital rule.
Remark: the first and second derivatives of the Taylor polynomial of order 2 at point $x = a$ coincide with those of $f$.

4.2.2 Second order approximation

The Taylor polynomial is the tangent parabola to $f$ (if $f''(a) \neq 0$). What is the Taylor polynomial good for if $f''(a) \neq 0$? In other words, what is the tangent parabola used for?

1. To know the relative position of the graph of $f$ with respect to the tangent line.
2. Also, if $f'(a) = 0$, to study local extrema by the sign of $f''(a)$.
   
   Let us assume that $f'(a) = 0, f''(a) \neq 0$. If the polynomial has a local extremum, $f$ does as well. Obviously, if the function does not have it, neither does the polynomial.
   
   See also section 4.3.
3. To obtain better approximations.

   **Example 4.2.3.** Find an approximated value of $\ln(0.9)$ and $\ln(1.2)$ using:
   
   a) the Taylor polynomial of $f(x) = \ln(1 + x)$ at $a = 0$: $\ln(1 + x) \approx x - x^2/2$; or
   
   b) the Taylor polynomial of $f(x) = \ln(x)$ at $a = 1$: $\ln(x) \approx (x - 1) - (x - 1)^2/2$

4.3 Second order optimality conditions

Let $f$ be a function of class $C^2$.

**Necessary conditions**

- $f(c)$ is a local minimum of $f \Rightarrow f''(c) \geq 0$;
- $f(c)$ is a local maximum of $f \Rightarrow f''(c) \leq 0$.

**Sufficient conditions**

Let $c$ be a critical point, $f'(c) = 0$.

- $f''(c) > 0 \Rightarrow f(c)$ is a (strict) local minimum of $f$;
- $f''(c) < 0 \Rightarrow f(c)$ is a (strict) local maximum of $f$.

**Example 4.3.1.** Let $f(x) = 4x^4 - \frac{8}{3}x^3 + 1$. We study local extrema with the first and second derivative. We have that $f'(x) = 16x^3 - 8x^2$ and $f''(x) = 48x^2 - 16$. Critical points are $x = 0$ and $x = \frac{1}{2}$. Since $f''(0) = 0$, we cannot conclude anything by the second derivative test. We have that, $f''(\frac{1}{2}) = \frac{48}{4} - \frac{16}{2} = 12 - 8 = 4 > 0$, therefore $\frac{1}{2}$ is a local minimizer of $f$. In order to tell what type of point 0 is, we can resort to the first derivative test, since $f'(-1) < 0, f'(1/4) < 0$, it follows that $f$ decreases when $x < \frac{1}{2}$ and, therefore, $x = 0$ it is neither a local maximizer nor a local minimizer.
Example 4.3.2. Let \( f(x) = 4x^4 - 2x^2 + 1 \), so \( f'(x) = 16x^3 - 4x \) and \( f''(x) = 48x^2 - 4 \). Can point \( c = 0 \) be a local minimizer of \( f \)? No, since \( f''(0) = -4 < 0 \). Is \( c = 0 \) a local maximizer of \( f \)? Yes, since it is a critical point, \( f'(0) = 0 \) and \( f''(0) \) is negative as we have computed above. Does \( f \) have other extremal points? Let us find all its critical points: \( f'(x) = 0 \) if and only if \( x = 0, x = \pm \frac{1}{2} \). Now, \( f''(\pm \frac{1}{2}) = 8 > 0 \), thus both \( \frac{1}{2} \) and \( -\frac{1}{2} \) are local minimizers.

4.4 Convexity and points of inflection of a function

Assume that the function \( f \) has a finite derivative at every point of the interval \((a,b)\). Then, at every point in \((a,b)\) the graph of the function has a tangent which is nonparallel to the \(y\)-axis.

Definition 4.4.1. The function \( f \) is said to be convex (concave) in the interval \((a,b)\) if, within \((a,b)\), the graph of \( f \) lies not lower (not higher) than any tangent.

Theorem 4.4.2 (Characterization of the convexity or concavity by the derivative).

1. \( f \) is convex on the interval \( I \) if and only if its derivative increases on \( I \).
2. \( f \) is concave on the interval \( I \) if and only if its derivative decreases on \( I \).

Theorem 4.4.3 (A sufficient condition for convexity/concavity). If \( f \) has second derivative in the interval \((a,b)\) and \( f''(x) \geq 0 \) (\( f''(x) \leq 0 \)) for every \( x \in (a,b) \), then \( f \) is convex (concave) in \((a,b)\).

Theorem 4.4.4 (Global Extrema of concave/convex functions).

1. If \( f \) is convex on \( I \) and \( c \) is a critical point of \( f \), then \( c \) is a global minimizer of \( f \) on \( I \).
2. If \( f \) is concave on \( I \) and \( c \) is a critical point of \( f \), then \( c \) is a global maximizer of \( f \) on \( I \).

Definition 4.4.5. A point \( c \) is a point of inflection of the function \( f \) if at this point the function changes the curvature, from convex to concave or from concave to convex.

Theorem 4.4.6 (A necessary condition for inflection point). If \( f \) has an inflection point at \( c \) and \( f'' \) is continuous in an interval around \( c \), then \( f''(c) = 0 \).

Theorem 4.4.7 (A sufficient condition for inflection point). If \( f'' \) exists in an interval around \( c \), with \( f''(c) = 0 \), and the signs of \( f'' \) are different on the left and on the right of the point \( c \), then \( c \) is an inflection point of \( f \).

Example 4.4.8. Find the intervals of concavity/convexity of \( f(x) = (x + 6)^3(x - 2) \), and the possible inflection points.
SOLUTION: The domain of \( f \) is the whole real line, and the function is continuous.

\[
f'(x) = 3(x + 6)^2(x - 2) + (x + 6)^3 = (x + 6)^2(3(x - 2) + (x + 6)) = (x + 6)^2(4x),
\]

\[
f''(x) = 8(x + 6)x + 4(x + 6)^2 = 4(x + 6)(2x + (x + 6)) = 12(x + 6)(x + 2).
\]

Hence, \( f'' \geq 0 \) in the region \( x \geq -2 \) and in the region \( x \leq -6 \), and \( f'' \leq 0 \) in the complement set, \([-6,-2]\). We conclude that \( f \) is convex in the interval \((-\infty,-6]\) and in the interval \([-2,\infty)\), and it is concave in the interval \([-6,-2]\). Obviously, \(-6\) and \(-2\) are inflection points.

4.5 Applications of the derivative to revenue, cost and profit functions of a firm

Recall the concepts of revenue function \( R \), cost function \( C \), and profit function \( \Pi \) of a firm given in the lesson about continuity of functions. Also remember that \( P(x) \) represents the market inverse demand function, and \( x \) is the quantity of the commodity produced and sold by the firm. We consider three different optimization problems.

**Owner’s Problem:** to maximize profits

\[
\max \Pi(x) \quad \text{subject to } x \text{ being feasible}.
\]

**Sales Manager Problem:** to maximize revenue

\[
\max R(x) \quad \text{subject to } x \text{ being feasible}.
\]

**Production Manager Problem:** to minimize average cost

\[
\min \frac{C(x)}{x} \quad \text{subject to } x > 0 \text{ being feasible}.
\]

Let

\[
P(x) = A - Bx;
\]

\[
C(x) = c + ax + bx^2,
\]

where \( A, B, b, c \) are non-negative, with \( A > 0, B > 0, b > 0 \) and \( A \geq a \). We have

\[
R(x) = xP(x) = x(A - Bx);
\]

\[
\Pi(x) = R(x) - C(x) = x(A - Bx) - (c + ax + bx^2);
\]

\[
\overline{C}(x) = \frac{C(x)}{x} = \frac{c}{x} + a + bx.
\]

Suppose that there is no production constraints, so that the good can be produced in any quantity.
• Owner’s Problem.

\[ \Pi'(x) = A - 2Bx - a - 2bx = 0 \Rightarrow x^* = \frac{A - a}{2(B + b)}. \]

Since

\[ \Pi''(x) = -2(B + b) < 0, \]

the profit function is strictly concave, thus \( x^* \) maximizes profits (unique global maximum).

• Sales Manager Problem.

\[ R'(x) = A - 2Bx = 0 \Rightarrow x^{**} = \frac{A}{2B}. \]

Since

\[ R''(x) = -2B < 0, \]

the revenue function is strictly concave, thus \( x^{**} \) maximizes revenue (unique global maximum).

• Production Manager Problem.

\[ C'(x) = -\frac{c}{x^2} + b = 0 \Rightarrow x^{***} = \sqrt[3]{\frac{c}{b}}. \]

Since

\[ C''(x) = \frac{2c}{x^3} > 0, \]

the average cost function is strictly convex, thus \( x^{***} \) minimizes average cost (unique global minimum).