Chapter 2

Limits and continuity of functions of one variable

2.1 Limits

To determine the behavior of a function \( f \) as \( x \) approaches a finite value \( c \), we use the concept of limit. We say that the limit of \( f \) is \( L \), and write \( \lim_{x \to c} f(x) = L \), if the values of \( f \) approaches \( L \) when \( x \) gets closer to \( c \).

**Definición 2.1.1.** (Limit when \( x \) approach a finite value \( c \)). We say that \( \lim_{x \to c} f(x) = L \) if for any small positive \( \epsilon \), there is a positive \( \delta \) such that

\[
|f(x) - L| < \epsilon
\]

whenever \( 0 < |x - c| < \delta \).

We can split the above definition in two parts, using one-sided limits.

**Definición 2.1.2.**

1. We say that \( L \) is the limit of \( f \) as \( x \) approaches \( c \) from the right, \( \lim_{x \to c^+} f(x) = L \), if for any small positive \( \epsilon \), there is a positive \( \delta \) such that

\[
|f(x) - L| < \epsilon
\]

whenever \( 0 < x - c < \delta \).

2. We say that \( L \) is the limit of \( f \) as \( x \) approaches \( c \) from the left, \( \lim_{x \to c^-} f(x) = L \), if for any small positive \( \epsilon \), there is a positive \( \delta \) such that

\[
|f(x) - L| < \epsilon
\]

whenever \( 0 < c - x < \delta \).

**Teorema 2.1.3.** \( \lim_{x \to c} f(x) = L \) if and only if

\[
\lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.
\]
We can also wonder about the behavior of the function \( f \) when \( x \) approaches \(+\infty\) or \(-\infty\).

**Definición 2.1.4.** (Limits when \( x \) approaches \( \pm\infty \))

1. \( \lim_{x \to +\infty} f(x) = L \) if for any small positive \( \epsilon \), there is a positive value of \( x \), call it \( x_1 \), such that
   \[ |f(x) - L| < \epsilon \]
   whenever \( x > x_1 \).

2. \( \lim_{x \to -\infty} f(x) = L \) if for any small positive \( \epsilon \), there is a negative value of \( x \), call it \( x_1 \), such that
   \[ |f(x) - L| < \epsilon \]
   whenever \( x < x_1 \).

If the absolute values of a function become arbitrarily large as \( x \) approaches either a finite value \( c \) or \( \pm\infty \), then the function has no finite limit \( L \) but will approach \(-\infty\) or \(+\infty\). It is possible to give the formal definitions. For example, we will say that \( \lim_{x \to c} f(x) = +\infty \) if for any large positive number \( M \), there is a positive \( \delta \) such that
\[ f(x) > M \]
whenever \( 0 < |x - c| < \delta \). Please, complete the remaining cases.

**Nota 2.1.5.** Note that it could be \( c \in D(f) \), so \( f(c) \) is well defined, but \( \lim_{x \to c} f(x) \) does not exits or \( \lim_{x \to c} f(x) \neq f(c) \). Consider for instance the function \( f \) that is equal to 1 for \( x \neq 0 \), but \( f(0) = 0 \). Then clearly the limit of \( f \) at 0 is \( 1 \neq f(0) \).

**Ejemplo 2.1.6.** Consider the following limits.

1. \( \lim_{x \to 0} x^2 - 2x + 7 = 31 \).

2. \( \lim_{x \to \pm\infty} x^2 - 2x + 7 = \infty \), because the leading term in the polynomial gets arbitrarily large.

3. \( \lim_{x \to +\infty} x^3 - x^2 = \infty \), because the leading term in the polynomial gets arbitrarily large for large values of \( x \), but \( \lim_{x \to -\infty} x^3 - x^2 = -\infty \) because the leading term in the polynomial gets arbitrarily large in absolute value, and negative.

4. \( \lim_{x \to \pm\infty} \frac{1}{x} = 0 \), since for \( x \) arbitrarily large in absolute value, \( 1/x \) is arbitrarily small.

5. \( \lim_{x \to 0} \frac{1}{x} \) does not exists. Actually, the one–sided limits are:
   \[ \lim_{x \to 0^+} \frac{1}{x} = +\infty \]
   \[ \lim_{x \to 0^-} \frac{1}{x} = -\infty \]
The right limit is infinity because \( 1/x \) becomes arbitrarily large when \( x \) is small and positive. The left limit is minus infinity because \( 1/x \) becomes arbitrarily large in absolute value and negative, when \( x \) is small and negative.

6. \( \lim_{x \to +\infty} x \sin x \) does not exist. As \( x \) approaches infinity, \( \sin x \) oscillates between 1 and –1. This means that \( x \sin x \) changes sign infinitely often when \( x \) approaches infinity, whilst taking arbitrarily large absolute values. The graph is shown below.

7. Consider the function \( f(x) = \begin{cases} x^2, & \text{if } x \leq 0; \\ -x^2, & \text{if } 0 < x \leq 1; \\ x, & \text{if } x > 1. \end{cases} \) \( \lim_{x \to 1} f(x) \) does not exist since the one–sided limits are different.

\[
\begin{align*}
\lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x = 1, \\
\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} -x^2 = -1.
\end{align*}
\]

8. \( \lim_{x \to 0} \frac{|x|}{x} \) does not exist, because the one–sided limits are different.

\[
\begin{align*}
\lim_{x \to 0^+} \frac{|x|}{x} &= \lim_{x \to 0^+} \frac{x}{x} = 1, \\
\lim_{x \to 0^-} \frac{|x|}{x} &= \lim_{x \to 0^-} \frac{-x}{x} = -1 \quad (\text{when } x \text{ is negative, } |x| = -x).
\end{align*}
\]

In the following, \( \lim f(x) \) refer to the limit as \( x \) approaches \(+\infty\), \(-\infty\) or a real number \( c \), but we never mix different type of limits.

### 2.1.1 Properties of limits

\( f \) and \( g \) are given functions and we suppose that all the limits below exist; \( \lambda \in \mathbb{R} \) denotes an arbitrary scalar.

1. **Product by a scalar:** \( \lim \lambda f(x) = \lambda \lim f(x) \).

2. **Sum:** \( \lim (f(x) + g(x)) = \lim f(x) + \lim g(x) \).
3. **Product:** \( \lim f(x)g(x) = (\lim f(x))(\lim g(x)) \).

4. **Quotient:** If \( \lim g(x) \neq 0 \), then \( \lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} \).

**Teorema 2.1.7** (Squeeze Theorem). Assume that the functions \( f, g \) and \( h \) are defined around the point \( c \), except, maybe, for the point \( c \) itself, and satisfy the inequalities

\[
g(x) \leq f(x) \leq h(x).
\]

Let \( \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \). Then

\[
\lim_{x \to c} f(x) = L.
\]

**Ejemplo 2.1.8.** Show that \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0 \).

**Solution:** We use the theorem above with \( g(x) = -|x| \) and \( h(x) = |x| \). Notice that for every \( x \neq 0 \), \(-1 \leq \sin (1/x) \leq 1\) thus, when \( x > 0 \)

\[-x \leq x \sin (1/x) \leq x,\]

and when \( x < 0 \)

\[x \leq x \sin (1/x) \leq -x.\]

These inequalities mean that \(-|x| \leq x \sin (1/x) \leq |x|\). Since

\[
\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0,
\]

we can use the theorem above to conclude that \( \lim_{x \to 0} x \sin \frac{1}{x} = 0 \).

**2.1.2 Techniques for evaluating** \( \lim \frac{f(x)}{g(x)} \)

1. Use the property of the quotient of limits, if possible.

2. If \( \lim f(x) = 0 \) and \( \lim g(x) = 0 \), try the following:

   (a) Factor \( f(x) \) and \( g(x) \) and reduce \( \frac{f(x)}{g(x)} \) to lowest terms.

   (b) If \( f(x) \) or \( g(x) \) involves a square root, then multiply both \( f(x) \) and \( g(x) \) by the conjugate of the square root.

**Ejemplo 2.1.9.**

\[
\lim_{x \to 3} \frac{x^2 - 9}{x + 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{x + 3} = \lim_{x \to 3} (x - 3) = 0.
\]

\[
\lim_{x \to 0} \frac{1 - \sqrt{1 + x}}{x} = \lim_{x \to 0} \frac{1 - \sqrt{1 + x}}{x} \left( \frac{1 + \sqrt{1 + x}}{1 + \sqrt{1 + x}} \right) = \lim_{x \to 0} \frac{-x}{x(1 + \sqrt{1 + x})} = \lim_{x \to 0} \frac{-1}{1 + \sqrt{1 + x}} = -\frac{1}{2}.
\]
3. If \( f(x) \neq 0 \) and \( \lim g(x) = 0 \), then either \( \lim \frac{f(x)}{g(x)} \) does not exist or \( \lim \frac{f(x)}{g(x)} = +\infty \) or \( -\infty \).

4. If \( x \) approaches \( +\infty \) or \( -\infty \), divide the numerator and denominator by the highest power of \( x \) in any term of the denominator.

**Ejemplo 2.1.10.**

\[
\lim_{x \to \infty} \frac{x^3 - 2x}{-x^4 + 2} = \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{2}{x^3}}{-1 + \frac{2}{x^4}} = \frac{0 - 0}{-1 + 0} = 0.
\]

### 2.1.3 Exponential limits

Let the limit \( \lim_{x \to c} [f(x)]^{g(x)} \) be an indetermination. This happens if

- \( \lim_{x \to c} f(x) = 1 \) and \( \lim_{x \to c} g(x) = \infty (1^\infty) \).
- \( \lim_{x \to c} f(x) = 0 \) and \( \lim_{x \to c} g(x) = 0 (0^0) \).
- \( \lim_{x \to c} f(x) = \infty \) and \( \lim_{x \to c} g(x) = 0 (\infty^0) \).

Noting that

\[
\lim_{x \to c} [f(x)]^{g(x)} = \lim_{x \to c} e^{g(x) \ln f(x)} = e^{\lim_{x \to c} g(x) \ln f(x)},
\]

all cases are reduced to the indetermination \( 0 \cdot \infty \), since we have to compute the limit

\[
\lim_{x \to c} g(x) \ln f(x).
\]

In the first indetermination, \( 1^\infty \), it often helps to use the identity

\[
\lim_{x \to c} g(x) \ln f(x) = \lim_{x \to c} g(x)(f(x) - 1).
\]

since when \( x \) is close to 0, \( \ln(1 + x) \approx x \), or, \( \ln x \approx x - 1 \)

**Ejemplo 2.1.11.** \( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to \infty} e^{x \ln \left(1 + \frac{1}{x}\right)} = e^{x \frac{1}{2} x} = e. \)

**Ejemplo 2.1.12.** Let \( a, b > 0 \). Calculate \( \lim_{x \to \infty} \left(\frac{1 + ax}{2 + bx}\right)^x \).

If \( a > b \), then the basis function tends to \( a/b > 1 \), thus the limit is \( \infty \). If \( a < b \), then the basis function tends to \( a/b < 1 \), thus the limit is 0. When \( a = b \)

\[
\lim_{x \to \infty} \left(\frac{1 + ax}{2 + ax}\right)^x = e^{\lim_{x \to \infty} x \left(\frac{1 + ax}{2 + ax} - 1\right)} = e^{\lim_{x \to \infty} \frac{-x}{2 + ax}} = e^{-1/a}.
\]

### 2.1.4 Remarkable limit

Recall that

\[
\lim_{x \to 0} \frac{\tan x}{x} = 1.
\]

**Ejemplo 2.1.13.** Evaluate the following limits:

1. \( \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1. \)
2. \[ \lim_{x \to 0} \frac{\sin 3x}{x} \{z=3x\} = \lim_{z \to 0} \frac{\sin z}{z} = 3 \lim_{z \to 0} \frac{\sin z}{z} = 3. \]

2.2 Asymptotes

An *asymptote* is a line that the graph of a function approaches more and more closely until the distance between the curve and the line almost vanishes.

**Definición 2.2.1.** Let \( f \) be a function

1. The line \( x = c \) is a vertical asymptote of \( f \) if \( \lim_{x \to c^+} |f(x)| = \infty \) or \( \lim_{x \to c^-} |f(x)| = \infty \).

2. The line \( y = b \) is a horizontal asymptote of \( f \) if \( \lim_{x \to +\infty} f(x) = b \) or \( \lim_{x \to -\infty} f(x) = b \).

3. The line \( y = ax + b \) is an oblique asymptote of \( f \) if
   
   (a) \( \lim_{x \to +\infty} \frac{f(x)}{x} = a \) and \( \lim_{x \to +\infty} (f(x) - ax) = b \), or
   
   (b) \( \lim_{x \to -\infty} \frac{f(x)}{x} = a \) and \( \lim_{x \to -\infty} (f(x) - ax) = b \).

Notice that a horizontal asymptote is a particular case of oblique asymptote with \( a = 0 \).

**Ejemplo 2.2.2.** Determine the asymptotes of \( f(x) = \frac{(1 + x)^4}{(1 - x)^4} \).

**SOLUTION:** Since the denominator vanishes at \( x = 1 \), the domain of \( f \) is \( \mathbb{R} \setminus \{1\} \). Let us check that \( x = 1 \) is a vertical asymptote of \( f \):

\[ \lim_{x \to 1^\pm} \frac{(1 + x)^4}{(1 - x)^4} = +\infty \]

On the other hand

\[ \lim_{x \to +\infty} \frac{(1 + x)^4}{(1 - x)^4} = \lim_{x \to +\infty} \frac{(1/x + 1)^4}{(1/x - 1)^4} = 1 \]

hence \( y = 1 \) is a horizontal asymptote at \( +\infty \). In the same way, \( y = 1 \) is a horizontal asymptote at \( -\infty \). There is no other oblique asymptotes.

**Ejemplo 2.2.3.** Determine the asymptotes of \( f(x) = \frac{3x^3 - 2}{x^2} \).

**SOLUTION:** The domain of \( f \) is \( \mathbb{R} \setminus \{0\} \). Let us check that \( x = 0 \) is a vertical asymptote of \( f \).

\[ \lim_{x \to 0^\pm} \frac{3x^3 - 2}{x^2} = \lim_{x \to 0^\pm} \frac{3x - 2}{x^2} = \lim_{x \to 0^\pm} 3x - \lim_{x \to 0^\pm} \frac{2}{x^2} = -\infty. \]

Thus, \( x = 0 \) is a vertical asymptote of \( f \). On the other hand

\[ \lim_{x \to \pm\infty} \frac{3x^3 - 2}{x^2} = \lim_{x \to \pm\infty} \frac{3x - 2}{x^2} = \pm\infty \]
thus, there is no horizontal asymptote. Let us study now oblique asymptotes:

\[
a = \lim_{x \to \pm \infty} \frac{f(x)}{x} = \lim_{x \to \pm \infty} \frac{3x^3 - 2}{x^3} = \lim_{x \to \pm \infty} \left(3 - \frac{2}{x^3}\right) = 3, \\
b = \lim_{x \to \pm \infty} (f(x) - 3x) = \lim_{x \to \pm \infty} \left(\frac{3x^3 - 2}{x^2} - 3x\right) = \lim_{x \to \pm \infty} \left(-\frac{2}{x^2}\right) = 0.
\]

We conclude that \( y = 3x \) is an oblique asymptote both at \(+\infty\) and \(-\infty\).

### 2.3 Continuity

The easiest limits to evaluate are those involving continuous functions. Intuitively, a function is continuous if one can draw its graph without lifting the pencil from the paper.

**Definición 2.3.1.** A function \( f : \mathbb{R} \to \mathbb{R} \) is continuous at \( c \) if \( c \in D(f) \) and

\[
\lim_{x \to c} f(x) = f(c).
\]

Hence, \( f \) is **discontinuous at \( c \)** if either \( f(c) \) is undefined or \( \lim_{x \to c} f(x) \) does not exist or \( \lim_{x \to c} f(x) \neq f(c) \).

#### 2.3.1 Properties of continuous functions

Suppose that the functions \( f \) and \( g \) are both continuous at \( c \). Then the following functions are also continuous at \( c \).

1. **Sum.** \( f + g \).
2. **Product by a scalar.** \( \lambda f \), \( \lambda \in \mathbb{R} \).
3. **Product.** \( fg \).
4. **Quotient.** \( f/g \), whenever \( g(c) \neq 0 \).

#### 2.3.2 Continuity of a composite function

Suppose that \( f \) is continuous at \( c \) and \( g \) is continuous at \( f(c) \). Then, the composite function \( g \circ f \) is also continuous at \( c \).

#### 2.3.3 Continuity of elementary functions

A function is called **elementary** if it can be obtained by means of a finite number of arithmetic operations and superpositions involving basic elementary functions. The functions \( y = C = \text{constant}, y = x^a, y = e^x, y = \ln x, y = \cos x, y = \sin x, y = \tan x, y = \arctan x \) are examples of elementary functions. **Elementary functions are continuous in their domain.**

**Ejemplo 2.3.2.**
1. The function $f(x) = \sqrt{4 - x^2}$ is the composition of the functions $y = 4 - x^2$ and $f(y) = y^{1/2}$, which are elementary, thus $f$ is continuous in its domain, that is, in $D = [-2, +2]$.

2. The function $g(x) = \frac{1}{\sqrt{4 - x^2}}$ is the composition of function $f$ above and function $g(y) = 1/y$, thus it is elementary and continuous in its domain, $D(g) = (-2, +2)$.

### 2.3.4 Limit of a composite function

Let $f,g$ be functions from $\mathbb{R}$ to $\mathbb{R}$ and $c \in \mathbb{R}$. If $g$ is continuous at $L$ and $\lim_{x \to c} f(x) = L$, then

$$\lim_{x \to c} g(f(x)) = g(\lim_{x \to c} f(x)) = g(L).$$

**Ejemplo 2.3.3.** Show that $\lim_{x \to 1} \arctan \left( \frac{x^2 + x - 2}{3x^2 - 3x} \right) = \frac{\pi}{4}$.

**Solution:** The function $\tan^{-1}$ is continuous.

$$\lim_{x \to 1} \arctan \left( \frac{x^2 + x - 2}{3x^2 - 3x} \right) = \arctan \left( \lim_{x \to 1} \frac{x^2 + x - 2}{3x^2 - 3x} \right)$$

$$= \arctan \left( \lim_{x \to 1} \frac{(x - 1)(x + 2)}{3(x - 1)(x + 1)} \right)$$

$$= \arctan \left( \lim_{x \to 1} \frac{x + 2}{3x} \right)$$

$$= \arctan 1$$

$$= \frac{\pi}{4}.$$

**Ejemplo 2.3.4.** Evaluate the following limits:

- $\lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \ln(1 + x)^{1/x} = \ln \left( \lim_{x \to 0} (1 + x)^{1/x} \right) = \ln e = 1.$

  Notice that the function $\ln(\cdot)$ is continuous at $e$, thus we can apply 2.3.4.

- $\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{z \to 0} \frac{z}{\ln(1 + z)} = \ln a \left( \lim_{z \to 0} \frac{z}{\ln(1 + z)} \right) = \ln a.$

  We have used $z = a^x - 1$, so that $x = \ln (1 + z)/\ln a$, and the value of the limit above.

### 2.3.5 Continuity theorems

Continuous functions have interesting properties. We shall say that a function is continuous in the closed interval $[a, b]$ if it is continuous at every point $x \in [a, b]$.

**Teorema 2.3.5 (Bolzano’s Theorem).** If $f$ is continuous in $[a, b]$ and $f(a) \cdot f(b) < 0$, then there exists some $c \in (a, b)$ such that $f(c) = 0$. 

Ejemplo 2.3.6. Show that the equation \( x^3 + x - 1 = 0 \) admits a solution, and find it with an error less than 0.1.

Solution: With \( f(x) = x^3 + x - 1 \) the problem is to show that there exists \( c \) such that \( f(c) = 0 \). We want to apply Bolzano’s Theorem. First, \( f \) is continuous in \( \mathbb{R} \). Second, we identify a suitable interval \( I = [a, b] \). Notice that \( f(0) = -1 < 0 \) and \( f(1) = 1 > 0 \) thus, there is a solution \( c \in (0, 1) \).

Now, to find an approximate value for \( c \), we use a method of interval-halving as follows: consider the interval \([0, 0.5]\); \( f(0) = -1 < 0 \) and \( f(0.5) = \frac{1}{8} + \frac{1}{2} - 1 < 0 \) thus, \( c \in (0, 0.5) \). Let now the interval \([0.625, 0.75]\); \( f(0.625) \approx -0.13 \) and \( f(0.74) > 0 \) thus, \( c \in (0.625, 0.75) \). The solution is approximately \( c = 0.6875 \) with a maximum error of 0.0625.

Teorema 2.3.7 (Weierstrass’ Theorem). If \( f \) is continuous in \([a, b]\), then there exist points \( c, d \in [a, b] \) such that
\[
 f(c) \leq f(x) \leq f(d)
\]
for every \( x \in [a, b] \).

The theorem asserts that a continuous function attains over a closed interval a minimum \((m = f(c))\) and a maximum value \((M = f(d))\). The point \( c \) is called a global minimizer of \( f \) on \([a, b]\) and \( d \) is called a global maximizer of \( f \) on \([a, b]\).

Ejemplo 2.3.8. Show that the function \( f(x) = x^2 + 1 \) attains over the closed interval \([-1, 2]\) a minimum and a maximum value.

Solution: The graph of \( f \) is shown below.

We can see that \( f \) is continuous in \([-1, 2]\), actually \( f \) is continuous in \( \mathbb{R} \), and \( f \) attains the minimum value at \( x = 0 \), \( f(0) = 1 \), and the maximum value at \( x = 2 \), \( f(2) = 5 \).

Ejemplo 2.3.9. The assumptions in the Theorem of Weierstrass are essential.

• The interval is not closed, or not bounded.
  – Take \( I = (0, 1] \) and \( f(x) = 1/x \); \( f \) is continuous in \( I \), but it does not have global maximum.
- Take $I = [0, \infty)$ and $f(x) = 1/(1 + x)$; $f$ is continuous in $I$, but it does not have global minimum, since $\lim_{x \to \infty} f(x) = 0$, but $f(x) > 0$ is strictly positive for every $x \in I$.

- The function is not continuous. Take $I = [0, 1]$ and $f(x) = \begin{cases} x, & \text{if } 0 \leq x < 1; \\ 0, & \text{if } x = 1. \end{cases}$ $f$ has a global minimum at $x = 0$, but there is no global maximum since $\lim_{x \to 1} f(x) = 1$ but $f(x) < 1$ for every $x \in I$. 
