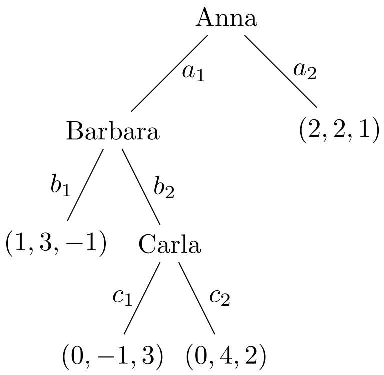
University Carlos III of Madrid **Departament of Economics**

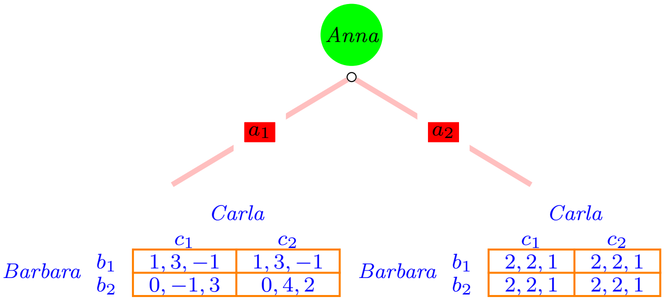
Game Theory: Problem set 2. Solutions.

Problem 1: Anna, Barbara and Clara are playing the following extensive form game,



(a) Write the game in its normal form.

Solution:

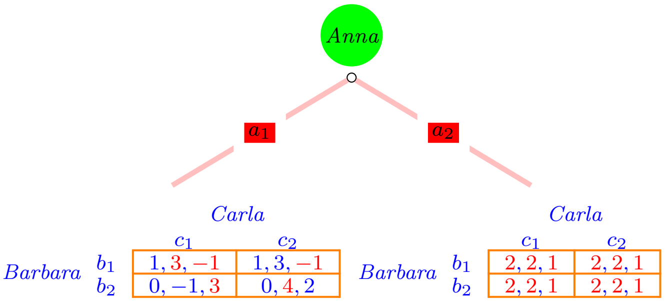


		Carla	
		c_1	c_2
Barbara	b_1	$1,3,-1$	$1,3,-1$
	b_2	$0,-1,3$	$0,4,2$

		Carla	
		c_1	c_2
Barbara	b_1	$2,2,1$	$2,2,1$
	b_2	$2,2,1$	$2,2,1$

(b) Find the pure strategy Nash equilibria of the game.

Solution: *The best replies of the players are*



		Carla	
		c_1	c_2
Barbara	b_1	$1,3,-1$	$1,3,-1$
	b_2	$0,-1,3$	$0,4,2$

		Carla	
		c_1	c_2
Barbara	b_1	$2,2,1$	$2,2,1$
	b_2	$2,2,1$	$2,2,1$

The NE are the following.

- (a_2, b_1, c_1).
- (a_2, b_1, c_2).
- (a_2, b_2, c_1).
- (a_2, b_2, c_2).

(c) Find the proper subgames of the game and write them in their normal forms.

Solution:

- *The game that starts with Barbara,*

		Carla	
		c_1	c_2
Barbara	b_1	$1, 3, -1$	$1, 3, -1$
	b_2	$0, -1, 3$	$0, 4, 2$

- *The game that starts with Carla,*

Carla	
c_1	c_2
$0, -1, 3$	$0, 4, 2$

(d) Find the Nash equilibria of such subgames.

Solution:

- *The NE of the game that starts with Barbara is* (b_1, c_1).
- *The NE of the game that starts with Barbara is* (c_1).

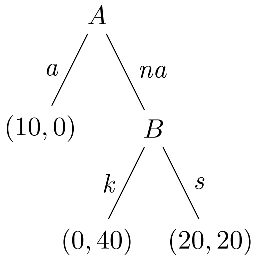
(e) Find the subgame perfect Nash equilibria of the game.

Solution: (a_2, b_1, c_1).

Problem 2: A firm's foundation has to choose whether to contribute to University A or to University B. Each University is interested only in the contributions it receives. The foundation announces that the contribution will be decided by the following mechanism. University A is proposed a contribution of \$10,000 for itself and \$0 for University B. If University A accepts, the contribution is assigned. Otherwise the Foundation increases the total contribution to \$40,000 and asks University B whether it wants to keep all the \$40,000 for itself or whether it prefers that the contribution is equally shared between the two Universities (\$20,000 to A and \$20,000 to B).

(a) Draw the extensive form of the game (i.e., the tree whose nodes represent when players play and whose branches represent their decisions).

Solution:



(b) Find the subgame perfect equilibrium of the game.

Solution: (a, k).

(c) Represent the game in its normal form (i.e., the matrix whose rows represent University A's strategies and whose columns represent University B's strategies, and the cells containing players' payoffs for any possible combination of strategies).

Solution: $S_A = \{a, na\}$, $S_B = \{k, s\}$. *The normal game for is the following.*

		B	
		k	s
A	a	$10, 0$	$10, 0$
	na	$0, 40$	$20, 20$

(d) Find the pure-strategy Nash equilibria. Has one of the players a dominant or a dominated strategy?

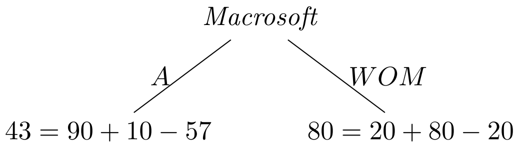
Solution: *The NE is (a, k). Strategy a is weakly dominated for player 2, but it is not strictly dominated.*

Problem 3: Macrosoft has developed a new video game and it wants to launch it on the market. In particular the choice is between using an impacting advertising campaign or leaving the promotion to the communication between buyers and potential buyers ('word of mouth'). Macrosoft knows that the video game will last only 2 years on the market (as in 2 years newer and better video games will be introduced). It also knows that the sales do not depend on marketing strategy, but the temporal distribution of the sales do. The advertising campaign produces high sales in the first year but lower in the second due to market saturation. Using the word of mouth strategy, sales will be lower in the first year and higher in the second year. The following table describes Macrosoft's income depending on the strategy it adopts.

	Advertising	Word of mouth
Macrosoft first year income	900,000	200,000
Macrosoft second year income	100,000	800,000
Marketing cost	570,000	200,000

(a) Draw the decision tree that represents Macrosoft problem and derive Macrosoft optimal decision.

Solution:

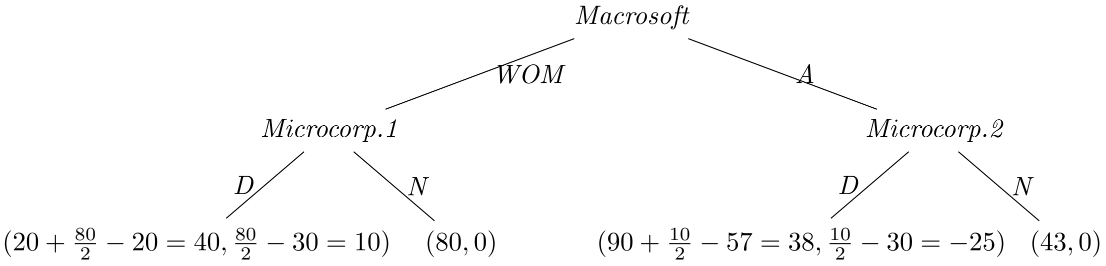


The optimal decision is WOM.

Assume now that a competitor, Microcorp, observes Macrosoft's strategy and can decide to develop a copy of the videogame in a year starting from the beginning of the promotion campaign. If Microcorp develops the copy, both firms share the market equally. The cost for Microcorp of developing the copy is 300.000.

(b) Draw the extensive form of the game(the 'game tree').

Solution:



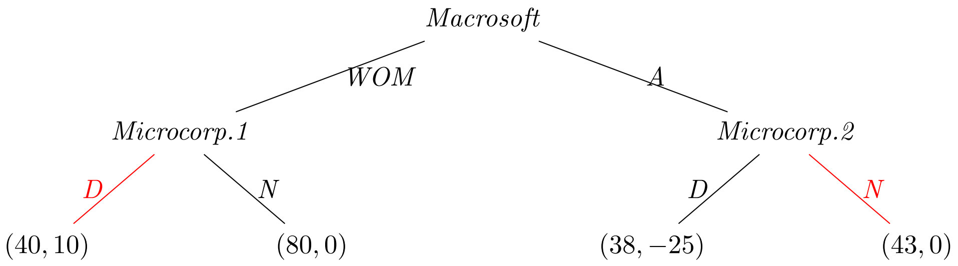
where

- *WOM = Word of mouth.*
- *A = Advertise.*
- *DN = Do not develop.*

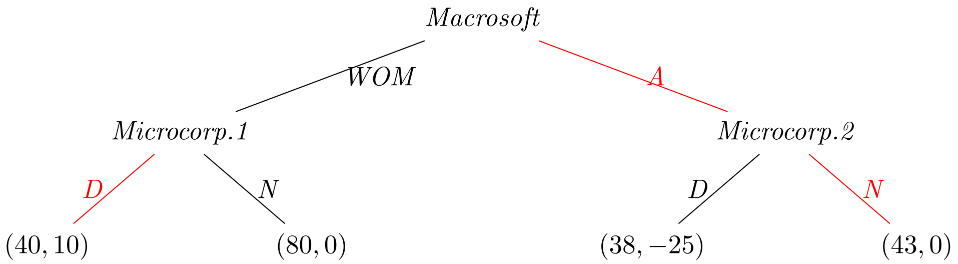
- *N = Develop.*

(c) Find the subgame perfect Nash equilibria of the game.

Solution: *We will use the backwards induction procedure. We draw in red the best responses of the players. In the he first step, Microcorp decides D at the node Microcorp.1 and N at the node Microcorp.2.*



Given this, Macrosoft's best option is to choose A.



The SPNE is (A, DN).

(d) What is Macrosoft's strategy space?

Solution: $S_{Macrosoft} = \{WOM, A\}$.

(e) What is Microcorp's strategy space?

Solution: $S_{Microcorp} = \{DD, DN, ND, NN\}$.

(f) What is the normal form of the game?

Solution:

		Microcorp			
		DD	DN	ND	NN
Macrosoft	WOM	$40,10$	$40,10$	$80,0$	$80,0$
	A	$38,-25$	$43,0$	$38,-25$	$43,0$

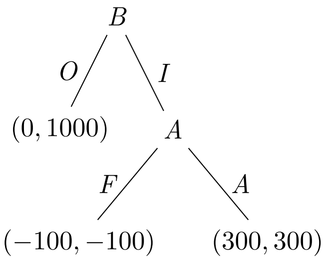
(g) Find the pure strategy Nash equilibria.

Solution: *The set of NE is (A, DN) and (WOM, DD). The NE (WOM, DD) is based on a non-credible threat.*

Problem 4: Companies A and B are considering developing a new commercial aircraft. Suppose that A is ahead in the development process and that B is considering whether to enter the competition. If B stays out, it earns zero profit and A, being the monopolist, earns a profit of \$1 billion. If B decides to enter and develop the rival airplane, then A has to decide whether to accommodate B peaceably, or to wage a price war. In the event of peaceful competition, each firm will make a profit of \$300 million. If there is a price war, each will lose \$100 million.

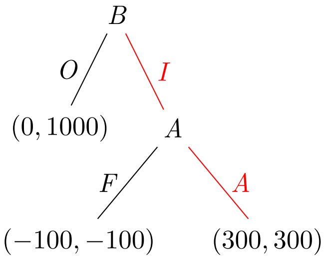
(a) Draw the game tree ('extensive form') for this game.

Solution:



(b) What is the subgame perfect Nash equilibrium?

Solution:



The SPNE is (I, A). *The payoffs are* $u_A = u_b = 300$.

(c) How many strategies does each player have?

Solution: $S_A = \{F, A\}$, $S_B = \{I, O\}$.

(d) Write the normal form of the game.

Solution:

		A	
		A	F
B	I	$300, 300$	$-100, -100$
	O	$0, 1000$	$0, 1000$

(e) Find all Nash equilibria of this game.

Solution: *Let us look for a NE of the form*

$$\begin{aligned} \sigma_1 &= xI + (1-x)O \\ \sigma_2 &= yA + (1-y)F \end{aligned}$$

We compute the expected utilities of the players

$$\begin{aligned} u_1(I, \sigma_2) &= 300y - 100(1-y) = 400y - 100 \\ u_1(O, \sigma_2) &= 0y + 0(1-y) = 0 \\ u_2(\sigma_1, A) &= 300x + 1000(1-x) = 1000 - 700x \\ u_2(\sigma_1, F) &= -100x + 1000(1-x) = 1000 - 1100x \end{aligned}$$

Note that

(a) $0 > 400y - 100$ *for* $0 < y < \frac{1}{4}$.

(b) $0 = 400y - 100$ *for* $y = \frac{1}{4}$.

(c) $0 < 400y - 100$ *for* $\frac{1}{4} < y \leq 1$.

Thus, we have that best reply of player 1 is

$$\mathrm{BR}_1(\sigma_2) = \begin{cases} x = 0 & \textit{if} \quad 0 \leq y < \frac{1}{4} \\ [0,1] & \textit{if} \quad y = \frac{1}{4} \\ x = 1 & \textit{if} \quad \frac{1}{4} < y \leq 1 \end{cases}$$

Note now that

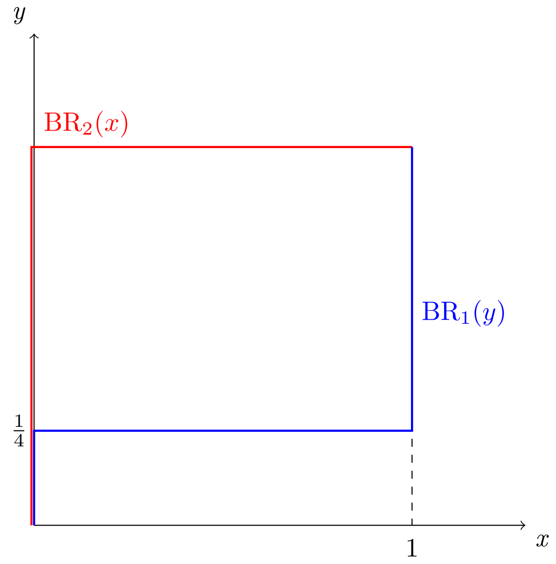
(a) $1000 - 700x > 1000 - 1100x$ *for* $x > 0$.

(b) $1000 - 700x = 1000 - 1100x$ *for* $x = 0$.

Thus, we have that best reply of player 2 is

$$\mathrm{BR}_2(\sigma_1) = \begin{cases} y = 1 & \textit{if} \quad 0 < x \leq 1 \\ [0,1] & \textit{if} \quad x = 0 \end{cases}$$

Graphically,



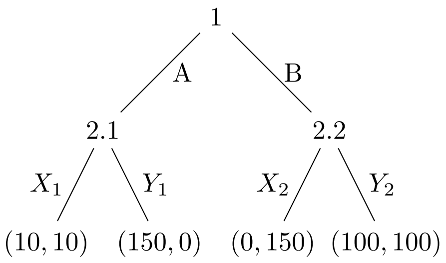
We see that there are the following NE,

- (O, F) *with payoffs* $u_A = 1000$, $u_B = 0$.
- (I, A) *with payoffs* $u_A = 300$, $u_B = 300$.
- $(O, yA + (1-y)F) \quad 0 \leq y \leq \frac{1}{4}$, *with payoffs* $u_A = 1000$, $u_B = 0$.

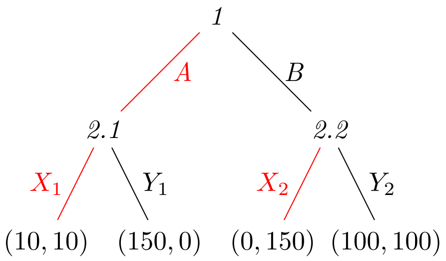
Only the NE (I, A) *is a SPNE. All the other NE are based on non-credible strategies.*

Problem 5: Find the subgame perfect Nash equilibria of the following games.

(a)

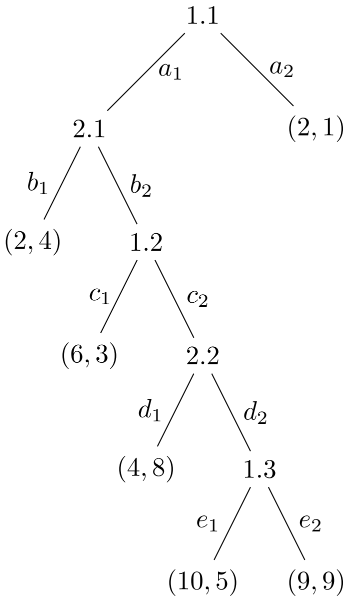


Solution:

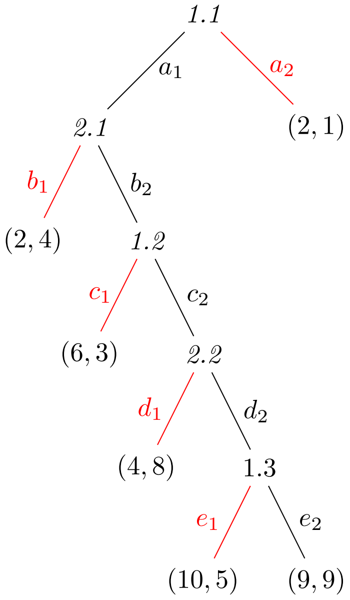


The SPNE is $(A, (X_1, X_2))$ *with payoffs* $u_1 = u_2 = 10$.

(b)

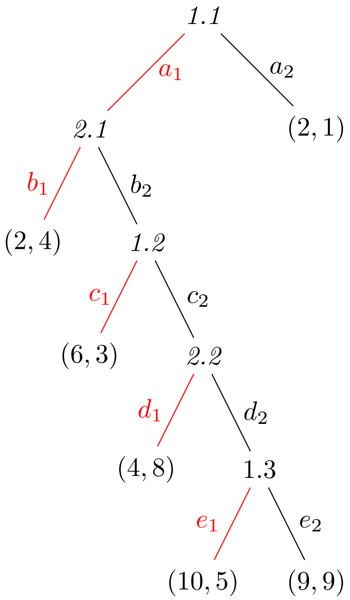


Solution: *One SPNE in pure strategies is* $((a_2, c_1, e_1), (b_1, d_1))$. *That is,*



with payoffs $u_1 = 2$, $u_2 = 1$.

The other SPNE in pure strategies is $((a_1, c_1, e_1), (b_1, d_1))$. *That is,*

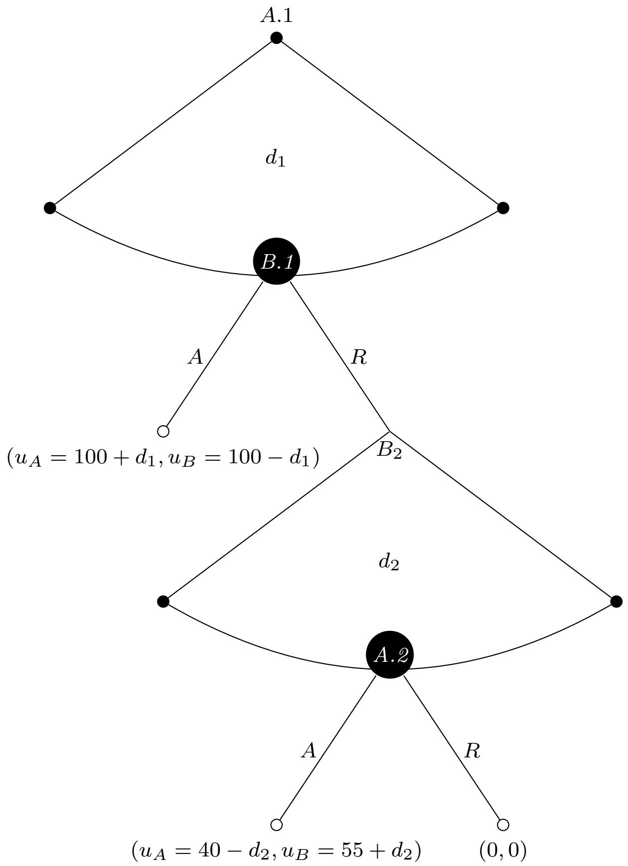


with payoffs $u_1 = 2$, $u_2 = 4$.

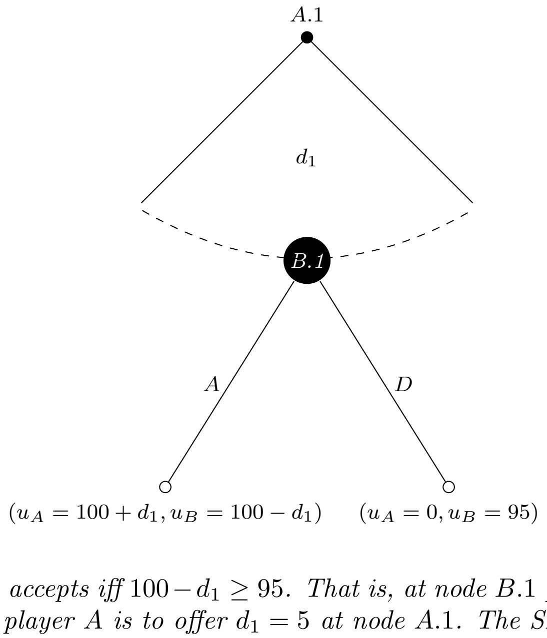
Problem 6: Suppose that two firms are negotiating an agreement. If the two parties reach an agreement immediately, each will have a gain of 100. If the two parties don't achieve an agreement immediately they will have a second and last chance of achieving an agreement in a month. If the agreement is reached in a month, firm A will have a gain of 40 and firm B will have a gain of 55. Negotiation works in the following way: In the first period Firm A demands a payment of firm B in order to sign the agreement. If firm B accepts, the agreement is reached, the payoff to firm A will be 100 plus the payment that firm B has accepted to make. And the payoff to firm B will be 100 minus the payment it has accepted to make. If firm B rejects the offer, it will make an offer to firm A in the second period with the offer consisting again of a demand of a payment in order to sign the agreement.

(a) How much does firm A demand of firm B in the first period in the subgame perfect Nash equilibrium of this game? (Hint: Solve the game by backward induction.)

Solution: *The extensive game form representation is the following.*



We compute the SPNE. At node A.2, firm A accepts iff $40 - d_2 \geq 0$. *That is, firm A accepts iff* $d_2 \leq 40$. *Given the best reply of firm A at node A.2, the best response of firm B at node B.2 is to offer* $d_2 = 40$. *Thus, we may assume that if we ever reach node B.2, firm B will offer* $d_2 = 40$ *and firm A accepts. The payoffs will be* $u_A = 0$, $u_B = 95$. *We replace this payoffs at node B.2*



Now, player B at node B.1 accepts iff $100 - d_1 \geq 95$. *That is, at node B.1 player B accepts iff* $d_1 \leq 5$. *The best response now for player A is to offer* $d_1 = 5$ *at node A.1. The SPNE is the following.*

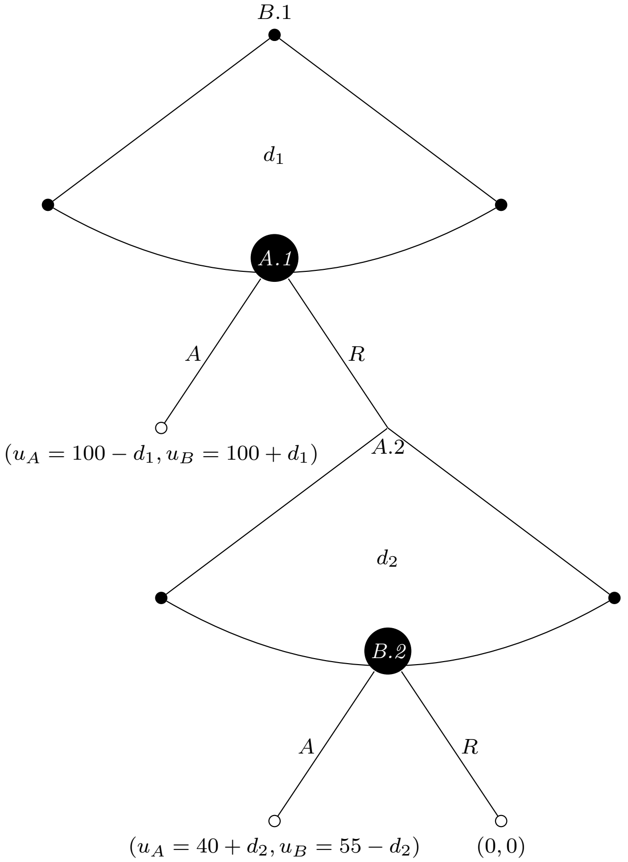
- *Node A.1:* $d_1 = 5$.
- *Node B.1: accept iff* $d_1 \leq 5$.

- *Node B.2:* $d_2 = 40$.
- *Node A.2: accept iff* $d_2 \leq 40$.

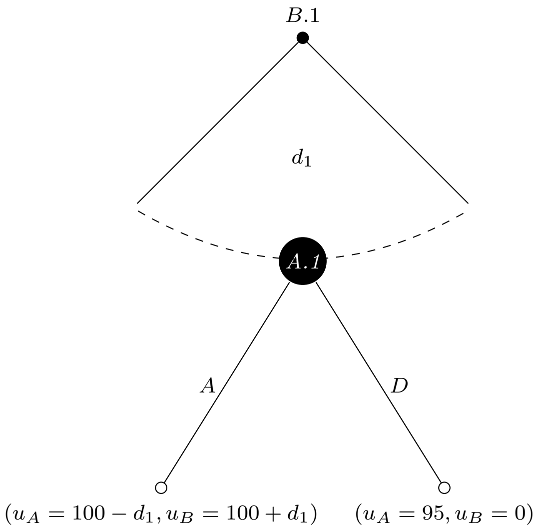
The payoffs are $u_A = 105$, $u_B = 95$.

(b) Suppose now that firm B made the offer in the first period and firm A in the second period. How much would firm B demand of firm A in the first period in the subgame perfect Nash equilibrium of this game?

Solution: *The roles of B and A are reversed now. The extension game form representation is the following. The extension game form representation is the following.*



We compute the SPNE. At node B.2, firm B accepts iff $55 - d_2 \geq 0$. *That is firm B accepts iff* $d_2 \leq 55$. *Given the best reply of firm B at node B.2, the best response of firm A at node A.2 is to offer* $d_2 = 55$. *Thus, we may assume that if we ever reach node A.2, firm A will offer* $d_2 = 55$ *and firm B accepts. The payoffs will be* $u_A = 95$, $u_B = 0$. *We replace this payoffs at node B.2*



Now, player A at node $A.1$ *accepts iff* $100 - d_1 \geq 95$. *That, is at node* $A.1$ *player A accepts iff* $d_1 \leq 5$. *The best response now for player B is to offer* $d_1 = 5$ *at node* $B.1$. *The SPNE is the following.*

- *Node* $B.1$*:* $d_1 = 5$.
- *Node* $A.1$*: accept iff* $d_1 \leq 5$.
- *Node* $A.2$*:* $d_2 = 40$.
- *Node* $B.2$*: accept iff* $d_2 \leq 40$.

The payoffs are $u_A = 95$, $u_B = 105$.

(c) If you worked for firm A would you rather have your firm be the first or the second to make the offer?

Solution: *Player A prefers to move first.*