# Chapter 4
# Repeated games

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## 1 Games repeated a finite number of times

Consider a simultaneous game $G$. Starting from $G$ (the **stage game**) we construct a dynamic game (**the repeated game**) $G_R$ which consists in playing the static game $G$ a number $n$ of times. That is, the repeated game $G_R$ consists of the same number of players as in $G$. But, now there are $t = 1, \ldots, n$ stages. At each stage $t = 2, \ldots, n$,

- Players are informed of what has been played in all the previous stages $1, \ldots, t-1$.
- The simultaneous game $G$ is played. Each player $i$ obtains a payoff $u_i^t$.

After the game $G$ has been played $n$ times, the total payoff that player $i$ receives in $G_R$ is

$$u_i = \sum_{t=1}^{n} u_i^t$$

We will be interested in studying the subgame perfect Nash equilibria of $G_R$. The following remarks will be useful.

*Observation* 1.1. At every stage of $G_R$ it starts a subgame of $G_R$. All of the subgames of $G_R$ are of this form.

*Observation* 1.2 (The one shot deviation principle). Let $G$ be a stage game. And let $G_R$ be the repeated game which consists in playing $G$ a finite number of times. A strategy profile is a SPNE of $G_R$ if and only if no player can gain by changing her action after any history, keeping the strategies of the other players as well as the rest of her strategy constant.

**Proposition 1.3.** *Let $G$ be a stage game. And let $G_R$ be the repeated game which consists in playing $G$ a finite number of times. Then, in the last stage of every SPNE of $G_R$, players play a NE of $G$.*

**Proposition 1.4.** *Let $G$ be a stage game. And let $G_R$ be the repeated game which consists in playing $G$ a finite number of times. The strategy profile in which at every stage, players play an unconditionally prescribed NE (that is the same NE at each stage of the repeated game) of the stage game $G$, constitutes a NE of the repeated game $G_R$.*

### 1.1 Repeated games repeated in which the stage game has a unique NE.

**Example 1.5.** Let us take a prisoner's dilemma as the stage game,

| | | Player 2 | |
|---|---|---|---|
| | | $D$ | $C$ |
| Player 1 | $D$ | $1, 1$ | $15, 0$ |
| | $C$ | $0, 15$ | $10, 10$ |

$G$ The stage game

There is a unique NE: $(D, D)$ with payoffs $u_1 = u_2 = 1$.

Consider the repeated game which consists in playing the above prisoner's dilemma (the stage game) 2 times. This dynamic game of imperfect information has a total of 5 subgames: One that starts at $t = 1$ (the whole game) and 4 (proper) subgames that start at $t = 2$.

The repeated game which consists in playing prisoner's dilemma (the stage game) 3 times has a total of 21 subgames: One that starts at $t = 1$ (the whole game), 4 (proper) subgames that start at $t = 2$ and 16 (proper) subgames that start at $t = 3$.

It is easy to show that the unique SPNE of the repeated game (regardless of the number of times it is played) consists in playing the unique NE of the stage game at each stage. This statement holds for all repeated games whose stage game has a unique NE.

**Proposition 1.6.** *If the stage game $G$ has a unique NE, then the repeated game which consists in playing $G$ a finite number of times has a unique SPNE which consists in playing at every stage the NE of $G$.*

## 1.2 Cooperation when the stage game has at least two NE.

**Example 1.7.** Let us consider now the following stage game $G$,

| | | Player 2 | | |
|---|---|---|---|---|
| | | $D$ | $C$ | $E$ |
| Player 1 | $D$ | $1, 1$ | $15, 0$ | $4, 0$ |
| | $C$ | $0, 15$ | $10, 10$ | $5, 1$ |
| | $E$ | $0, 4$ | $1, 6$ | $a, a$ |

The stage game

where $a$ is a parameter. Let us consider the case $a = 7$, that is,

| | | Player 2 | | |
|---|---|---|---|---|
| | | $D$ | $C$ | $E$ |
| Player 1 | $D$ | $1, 1$ | $15, 0$ | $4, 0$ |
| | $C$ | $0, 15$ | $10, 10$ | $5, 1$ |
| | $E$ | $0, 4$ | $1, 6$ | $7, 7$ |

The stage game $G$

Note that the stage game $G$ has two NE:

- $(D, D)$ with payoffs $u_1 = u_2 = 1$; and
- $(E, E)$ with payoffs $u_1 = u_2 = 7$.

Let us consider $G_R$ the repeated game which consist in playing $G$ two times. How many subgames does $G_R$ have?

We are going to show that the following strategy profile $S$ is a SPNE of $G_R$: Player $i = 1, 2$ plays the following,

- At $t = 1$, choose $C$;
- At $t = 2$, if $(C, C)$ was chosen at $t = 1$ then chose $E$. Otherwise, choose $D$.

Note that in the strategy profile $S$, players play a NE of $G$ at every subgame that starts at $t = 2$. Now, we replace each of the subgames that start at $t = 2$ with the payoffs obtained with the strategy profile $S$ in that subgame. We obtain the following normal form game.

| | | Player 2 | | |
|---|---|---|---|---|
| | | $D$ | $C$ | $E$ |
| Player 1 | $D$ | $1+1, 1+1$ | $15+1, 0+1$ | $1+1, 4+1$ |
| | $C$ | $0+1, 15+1$ | $10+7, 10+7$ | $0+1, 5+1$ |
| | $E$ | $4+1, 0+1$ | $10+1, 1+1$ | $7+1, 7+1$ |

that is

| | | Player 2 | | |
|---|---|---|---|---|
| | | $D$ | $C$ | $E$ |
| Player 1 | $D$ | $2, 2$ | $16, 1$ | $2, 5$ |
| | $C$ | $1, 16$ | $17, 17$ | $1, 6$ |
| | $E$ | $5, 1$ | $11, 2$ | $8, 8$ |

and we see that $(C, C)$ is a NE of this game. Thus, $S$ is a SPNE of $G_R$ with payoffs $u_1 = u_2 = 17$. What is the intuition behind this SPNE? Cooperation in the first stage is sustainable as long as there is a future in which there are rewards and punishments that can be used to provide incentives to the agents. That is,

$$\underbrace{10}_{\text{cooperation}} + \underbrace{a}_{\text{reward}} \geq \underbrace{15}_{\text{deviation}} + \underbrace{1}_{\text{punishment}}$$

Note that playing $(CC)$ at $t = 2$ cannot be part of any SPNE of $G_R$. Why?

## 2 Games repeated an infinite number of times

Consider a simultaneous game $G$. Starting from $G$ (the **stage game**). We construct a dynamic game (**the repeated game**) $G_R$ which consists in playing the static game $G$ an infinite number of times. That is, the repeated game $G_R$ consists of the same number of players as in $G$. But, now there are $t = 1, 2, \ldots$ stages. At each stage $t = 2, \ldots$

- Players are informed of what has been played in all the previous stages $1, \ldots, t-1$.
- The simultaneous game $G$ is played. Each player $i$ obtains a payoff $u_i^t$.

After the game $G$ has been played $n$ times, the total payoff that player $i$ receives in $G_R$ is

$$u_i = \sum_{t=1}^{n} u_i^t \delta^{t-1}$$

where $0 \leq \delta < 1$ is the **discount factor**.

*Observation* 2.1 (An alternative interpretation). Let $G$ be a stage game. We construct a repeated game $G_R$ as follows: At every stage $t = 1, \ldots$ the stage game $G$ is played. Next,

- with probability $1 - p$ the game $G_R$ ends,
- with probability $p$ the game $G_R$ continues to the stage $t + 1$.

We let $G_R$ be the repeated game which consists in playing $G$ infinitely many times. The game $G_R$ above is equivalent to the game which consists in playing the stage game $G$ infinitely many times, with a discount factor $\delta p$.

We will be interested in studying the subgame perfect Nash equilibria of $G_R$. The following remarks also apply to games repeated infinitely many times.

*Observation* 2.2. At every stage of $G_R$ it starts a subgame of $G_R$. All of the subgames of $G_R$ are of this form.

*Observation* 2.3. If $0 \leq \delta < 1$, then

$$1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta}$$

This will be useful in the computations below.

*Observation* 2.4 (The one shot deviation principle). Let $G$ be a stage game. And let $G_R$ be the repeated game which consists in playing $G$ infinitely many times. A strategy profile is a SPNE of $G_R$ if and only if no player can gain by changing her action after any history, keeping the strategies of the other players as well as the rest of her strategy constant.

**Proposition 2.5.** *Let $G$ be a stage game. And let $G_R$ be the repeated game which consists in playing $G$ infinitely many times. The strategy profile in which at every stage, players play an unconditionally prescribed NE (that is the same NE at each stage of the repeated game) of the stage game $G$ constitutes a NE of the repeated game $G_R$.*

On the other hand, unlike in games repeated a finite number of times we are going to show that if the stage game is repeated infinitely many times there are SPNE of the repeated game in which at no stage players play a NE of the stage game.

*Observation* 2.6 (A problem with games repeated infinitely many times). Repeated games in which the stage game is played infinitely many times, may have many SPNE.

## 2.1 Cooperation in the prisoner's dilemma

**Example 2.7.** Let us take the prisoner's dilemma used above as the stage game of the repeated game $G_R$ which is $G$ played infinitely many times, with discount factor $0 \leq \delta < 1$.

| | | Player 2 | |
|---|---|---|---|
| | | $D$ | $C$ |
| Player 1 | $D$ | $1, 1$ | $15, 0$ |
| | $C$ | $0, 15$ | $10, 10$ |

prisoner's dilemma

Let us consider the following strategy profile $T$ (trigger strategies). Player $i = 1, 2$ plays the following,

- At $t = 1$, choose $C$;
- At $t > 1$, if $(C, C)$ was chosen at $t = 1, \ldots, t-1$ then chose $C$. Otherwise, choose $D$.

For what values of $\delta$ is the strategy profile $T$ a NE of $G_R$? The utility obtained by any player under the strategy profile $T$ is

$$u_T = 10 + 10\delta + 10\delta^2 + \cdots 10\delta^t + 10\delta^{t+1} + 10\delta^{t+2} + 10\delta^{t+3} + \cdots$$

If that player deviates in stage $t+1$ (and chooses $D$), then her payoff would be

$$u_d = 10 + 10\delta + 10\delta^2 + \cdots 10\delta^t + 15\delta^{t+1} + \delta^{t+2} + \delta^{t+3} + \cdots$$

Thus, the strategy profile $T$ is a NE of $G_R$ if and only if

$$0 \leq u_T - u_d = \delta^{t+1}\left(-5 + 9\delta + 9\delta^2 + \cdots\right)$$

That is, the strategy profile $T$ is a NE of $G_R$ if and only if

$$5 \leq 9\delta(1 + \delta + \delta^2 + \cdots) = \frac{9\delta}{1-\delta}$$

i.e,. if $\delta \geq \frac{5}{14}$. Thus, we conclude that the strategy profile $T$ is a NE of $G_R$ if and only if $\delta \geq \frac{5}{14}$.

We have to check now that the strategy profile $T$ is also a NE in every subgame of $G_R$. There are two types of subgames starting at a stage $t$.

- Subgames in which at every stage $1, 2, \ldots, t-1$ the strategy profile $(C, C)$ was played . Then, the situation in the subgame that starts at this node is exactly like in the original game $G_R$, except that the payoffs are multiplied by $\delta^{t-1}$. The above argument shows that the strategy profile $T$ is also a NE of that subgame.
- Subgames in which at some stage $1, 2, \ldots, t-1$ the strategy profile $(C, C)$ was not played. In these subgames the strategy profile $T$ prescribes that $(C, C)$, a NE of the stage game $G$, is played in every stage. By Proposition 2.5, this is a SPNE of this subgame.

Therefore, if $\delta \geq \frac{5}{14}$, the strategy profile $T$ is a SPNE of $G_R$. Note that in this strategy profile at every stage of $G_R$ the players play $(C, C)$.

## 2.2 Cooperation in a Cournot model

**Example 2.8.** Consider a Cournot model of oligopoly with one homogenous product and two firms that compete in quantities $q_1$, $q_2$. Both have the same constant marginal cost $c$. The inverse demand function is given by $p(q) = a - q$ where $q = q_1 + q_2$. The stage game is a static Cournot game of competition. The utility of the firm $i = 1, 2$ is

$$\pi_i(q_1, q_2) = (a - c - q_1 - q_2)q_i$$

The best reply of firm $i = 1, 2$ is

$$\mathrm{BR}_i(q_j) = \frac{a - c - q_j}{2}$$

and the Cournot-Nash equilibrium is

$$q_1^* = q_2^* = \frac{a - c}{3}$$

with profits

$$\pi_1^* = \pi_2^* = \frac{(a - c)^2}{9}$$

with market price

$$p^* = \frac{a + 2c}{3}$$

If there was a unique firm in the market, the monopolist would choose the quantity that maximizes

$$(a - c - q)q$$

That is

$$q^m = \frac{a - c}{2}$$

with profits

$$\pi^m = \frac{(a - c)^2}{4}$$

and market price

$$p^m = \frac{a + c}{2}$$

Note that if both firms agree on

$$q_1^c = q_2^c = \frac{q^m}{2} = \frac{a - c}{4}$$

then, each gets a profit of

$$\pi_1^c = \pi_2^c = \frac{(a - c)^2}{8}$$

Both firms prefer this later outcome. The problem is that it is not a NE. If, say, firm 1 knows that firm 2 is going to produce $q_2^c = \frac{q^m}{2} = \frac{a-c}{4}$ then firm 1 should deviate and produce

$$\mathrm{BR}_1(q_2^c) = \frac{a - c - \frac{a-c}{4}}{2} = \frac{3(a - c)}{8}$$

with a profit of

$$\pi_1\left(\frac{3(a-c)}{8}, \frac{a-c}{4}\right) = \frac{9(a-c)^2}{64}$$

Suppose now that both firms repeat the above stage game infinitely many times with discount factor $0 \le \delta < 1$. Let $G_R$ be the resulting repeated game. Let us consider the following (trigger strategies) strategy profile $T$. Firm $i = 1, 2$ choose the following $q_i$,

- At $t = 1$, choose
$$q_i = \frac{q^m}{2} = \frac{a-c}{4}$$
- At $t > 1$, if
$$q_1 = q_2 = \frac{a-c}{4}$$
was chosen at $t = 1, \ldots, t-1$ then chose
$$q_i = \frac{q^m}{2} = \frac{a-c}{4}$$
Otherwise, choose
$$q_i = q_i^* = \frac{a-c}{3}$$

For what values of $\delta$ is the strategy profile $T$ is a NE of $G_R$. The utility obtained by any player under the strategy profile $T$ is

$$\begin{aligned} u_T &= \frac{(a-c)^2}{8} + \frac{(a-c)^2}{8}\delta + \frac{(a-c)^2}{8}\delta^2 + \cdots \frac{(a-c)^2}{8}\delta^t + \frac{(a-c)^2}{8}\delta^{t+1} + \frac{(a-c)^2}{8}\delta^{t+2} + \frac{(a-c)^2}{8}\delta^{t+3} + \cdots \\ &= (a-c)^2\left(\frac{1}{8} + \frac{\delta}{8} + \frac{\delta^2}{8} + \cdots \frac{\delta^t}{8} + \frac{\delta^{t+1}}{8} + \frac{\delta^{t+2}}{8} + \frac{\delta^{t+3}}{8} + \cdots\right) \end{aligned}$$

If that firm $i = 1, 2$ deviates in stage $t+1$, then its best option is to deviate to

$$\mathrm{BR}_i(q_2^c) = \frac{3(a-c)}{8}$$

with a profit of

$$\frac{9(a-c)^2}{64}$$

After that, in period $t+2$ the firm $j$ that is following the $T$ strategy profile, will switch to the Cournot-Nash equilibrium. Hence at period $t+2$ the best action for firm $i$ is to swith to the Cournot-Nash equilibrium with profit

$$\frac{(a-c)^2}{9}$$

Thus, if firm $i = 1, 2$ deviates in stage $t+1$, the maximum payoff it can obtain is

$$\begin{aligned} u_d &= \frac{(a-c)^2}{8} + \frac{(a-c)^2}{8}\delta + \frac{(a-c)^2}{8}\delta^2 + \cdots \frac{(a-c)^2}{8}\delta^t + \frac{9(a-c)^2}{64}\delta^{t+1} + \frac{(a-c)^2}{9}\delta^{t+2} + \frac{(a-c)^2}{9}\delta^{t+3} + \cdots \\ &= (a-c)^2\left(\frac{1}{8} + \frac{\delta}{8} + \frac{\delta^2}{8} + \cdots \frac{\delta^t}{8} + \frac{9\delta^{t+1}}{64} + \frac{\delta^{t+2}}{9} + \frac{\delta^{t+3}}{9} + \cdots\right) \end{aligned}$$

Thus, the strategy profile $T$ is a NE of $G_R$ if and only if

$$0 \le u_T - u_d = \delta^{t+1}(a-c)^2\left(\frac{1}{8} - \frac{9}{64} + \frac{\delta}{72(1-\delta)}\right) = \delta^{t+1}(a-c)^2\left(-\frac{1}{64} + \frac{\delta}{72(1-\delta)}\right)$$

That is, the strategy profile $T$ is a NE of $G_R$ if and only if

$$\frac{1}{64} \le \frac{\delta}{72(1-\delta)}$$

i.e,. if $\delta \geq \frac{9}{17}$. Thus, we conclude that the strategy profile $T$ is a NE of $G_R$ if and only if $\delta \geq \frac{9}{17}$.

We have to check now that the strategy profile $T$ is also a NE in every subgame of $G_R$. There are two types of subgames starting at a stage $t$.

- Subgames in which at every stage $1, 2, \ldots, t-1$ it was played $(q^m, q^m)$. Then, the situation in the subgame that starts at this node is exactly like in the original game $G_R$, except that the payoffs are multiplied by $\delta^{t-1}$. So, the above argument shows that the strategy profile $T$ is also a NE of that subgame.
- Subgames in which at some stage $1, 2, \ldots, t-1$ the strategy profile

  $$\left(\frac{q^m}{2}, \frac{q^m}{2}\right)$$

  was not played. In these subgames the strategy profile $T$ prescribes that $(q^*, q^*)$, a NE of the stage game $G$, is played. By Proposition 2.5, this is a SPNE of this subgame.

Therefore, if $\delta \geq \frac{9}{17}$, the strategy profile $T$ is a SPNE of $G_R$. Note that in this strategy profile, the players cooperate at every stage of the game $G_R$.