Chapter 1
Introduction to Game Theory. Normal Form Games

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1 Normal Form Games

A normal form Game G consists of three elements $G = (I, S, u)$; where

1. $I = \{1, 2, \dots, n\}$ is the set of decision makers, the **players**.
2. $S = S_1, \times S_2 \times \dots \times S_n$ describes the feasible actions of the players (their **strategies**). The players choose a strategy $s = (s_1, s_2, \dots, s_n) \in S$ **simultaneously**. An outcome is obtained.
3. $u = (u_1, u_2, \dots, u_n)$ describes the satisfaction (the **payoffs**) of the players with the outcomes. For each $i = 1, 2, \dots, n$, $u_i : S \to \mathbb{R}$.

Observation 1.1. In the course, almost all the time $I = 2$.

Example 1.2 (Prisoners' Dilemma)**.** Two suspects are arrested by the police and kept in separated cells. They cannot communicate with each other. The police have enough evidence to convince both of them of crime A. However, the police 'know' that they have committed a much more serious crime, crime B. Unfortunately, the police lack the evidence to convict them of crime B. The police interrogate each of them separately and offers the possibility of confessing (C) crime B and implicating the other partner. Thus, each of the suspects has to decide wether to confess crime B and implicate his partner or remain silent (N). If one of them confesses and the other does not, then the one that has confessed goes free and his partner is sentenced to 6 years in prison. If both of them confess, they are sentenced to 5 years in prison each (the judge takes into account that they have collaborated). If both of them remain silent, the police use the evidence they have to convict them for crime A and they are both sentenced to 1 year in prison each.

Why does this fit into the framework of a normal form game? Because,

1. The **players** are the suspects. $I = \{1, 2\}$.
2. Their feasible actions are to confess (C) or remain silent (N). Thus, $S_1 = S_2 = \{C, N\}$ describe their **strategies**.
3. The **payoffs** may be represented by

$$\begin{array}{rclcrcl} u_1(C,C) & = & -5 & \quad & u_2(C,C) & = & -5 \\ u_1(C,N) & = & 0 & & u_2(C,N) & = & -6 \\ u_1(N,C) & = & -6 & & u_2(N,C) & = & 0 \\ u_1(N,N) & = & -1 & & u_2(N,N) & = & -1 \end{array}$$

The above is a bit cumbersome. From now we will use the following notation.

		Suspect 2: N	Suspect 2: C
Suspect 1	N	-1, -1	$-6, 0$
	C	$0, -6$	$-5, -5$

The prisoner's dilemma

What do you think it will happen?

Example 1.3 (Advertising)**.** Two firms operate in a market. Presently, each one of them earns \$50 million from its customers. Now both have to decide wether to advertise (A) their product or not (N). Advertising costs each firm \$20 million. If one firm advertises and the other does not, the former captures \$30 million from the latter. If both firms advertise, they both earn \$50 million.

The above situation can be described as a normal form game with

1. The **players** are the firms. $I = \{1, 2\}$.
2. Their feasible actions are to advertise (A) or not to advertise (N). Thus $S_1 = S_2 = \{A, N\}$ describe their **strategies**.
3. The **payoffs** (profits) may be represented by

		Firm 2	
		N	A
Firm 1	N	$50, 50$	$20, 60$
	A	$60, 20$	$30, 30$

Advertising

Shi Qi: 'The impact of Advertising Regulation on Industry. The cigarrette Advertising Ban of 1971', Rand Journal (2013).

Example 1.4 (Tourists and Natives)**.** Two bars compete for customers. They can charge prices \$2, \$4 or \$5 for coffee. 6000 tourists choose a bar randomly. 4000 natives choose the bar with the lowest price.

We write this as a normal form game.

1. The **players** are bar 1 and bar 2. $I = \{1, 2\}$.
2. Their feasible actions are the prices). Thus the **strategies** are $S_1 = S_2 = \{2, 4, 5\}$.
3. The **payoffs** (profits) may be represented by

		Bar 2		
		2	4	5
	2	$10, 10$	$14, 12$	$14, 15$
Bar 1	4	$12, 14$	$20, 20$	$28, 15$
	5	$15, 14$	$15, 28$	$25, 25$

Tourists and Natives

2 Equilibrium

What does Game Theory predict in each of the above situations? (Standard) Game Theory makes the following assumptions.

- Rationality.
  1. Players maximize their payoffs.
  2. Players can make all the necessary inferences and computations.
- Common knowledge.
  1. Every player knows the rules of the game.
  2. Every player knows that every other player knows (**??**).

3. Every player knows that every other player knows (**??**).
4. Etc.

Some notation: Fix a player $i = 1, 2, \ldots, n$. We write the strategies by

$$s = (s_1, s_2, \ldots, s_n) = (s_i, s_{-i})$$

where

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots s_n)$$

2.1 Dominant strategies and Dominance Solvability.

Definition 2.1. Let a game $G = (I, S, u)$. Fix a player $i = 1, 2, \ldots, n$ and consider two actions $\bar{s}_i, \bar{\bar{s}}_i \in S_i$ for that player. We say that

- action $\bar{s}_i$ **weakly dominates** action $\bar{\bar{s}}_i$ (or that action $\bar{\bar{s}}_i$ is **weakly dominated** by action $\bar{s}_i$) for player i if

  $$u_i(\bar{s}_i, s_{-i}) \geq u_i(\bar{\bar{s}}_i, s_{-i}), \quad \text{for every } s_{-i} \in S_{-i}$$

  and

  $$u_i(\bar{s}_i, s_{-i}) > u_i(\bar{\bar{s}}_i, s_{-i}), \quad \text{for some } s_{-i} \in S_{-i}.$$

- item action $\bar{s}_i$ **strictly dominates** action $\bar{\bar{s}}_i$ (or that action $\bar{\bar{s}}_i$ is **strictly** by action $\bar{s}_i$) for player i if

  $$u_i(\bar{s}_i, s_{-i}) > u_i(\bar{\bar{s}}_i, s_{-i}), \quad \text{for every } s_{-i} \in S_{-i}.$$

Definition 2.2. Let a normal form game $G = (I, S, u)$. Fix a player $i = 1, 2, \ldots, n$. An action $s_i \in S_i$ is

- **weakly dominant** if weakly dominates every other action in S_i.
- **strictly dominant** if strictly dominates every other action in S_i.

Examples.

Iterated Elimination of Strictly Dominated Strategies (IESD):

Start with a normal form game G_0.

- Choose a player and remove all the strictly dominated strategies for that player. We obtain a new game G_1.
- Consider now the game G_1. Choose a player and remove all the strictly dominated strategies for that player. We obtain a new game G_2.
- Repeat the process until there are no strictly dominated strategies in the remaining game, say G_n.

The remaining strategies in the game G_n are called the **rationalizable** strategies.

Definition 2.3. A normal form game is **dominance solvable** if the IESD procedure yields a unique outcome.

Example 2.4 (Stug Hunt Game)**.** Two firms operate in a market. Presently, each one of them earns \$45 million from its customers. Now both have to decide wether to invest (I) all the money in R&D or not (N). However, R&D is successful only if both firms invest. If R&D is successful the the net profit for each firm is \$50 million.

The above situation can be described as a normal form game with

1. The **players** are the firms, $I = \{1, 2\}$.
2. Their feasible actions are to invest (I) or not to invest (N). Thus the **strategies** are $S_1 = S_2 = \{I, N\}$.
3. The **payoffs** (profits) may be represented by the following table.

Firm 2

Firm 1	I	N
I	$50, 50$	$0, 45$
N	$45, 0$	$45, 45$

Stug Hunt Game

Every strategy is rationalizable.

2.2 Nash Equilibrium in pure strategies.

Definition 2.5 (Best response for pure strategies)**.** Let a game $G = (I, S, u)$. Fix a player $i = 1, 2, \ldots, n$. Suppose he knows that all the other players are playing the strategy s_{-i}. We say an action $s_i \in S_i$ is a best response for player i to s_{-i} (and we write $s_i \in \mathrm{BR}_i(s_{-i})$ if

$$u_i(s_i, s_{-i}) \geq u_i(\bar{s}_i, s_{-i}), \quad \text{for every other } \bar{s}_i \in S_i$$

Definition 2.6 (pure strategy Nash Equilibrium)**.** Let a game $G = (I, S, u)$. A strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is a (pure strategy) Nash Equilibrium of G if for every player $i = 1, 2, \ldots, n$ we have that

$$s_i^* \in \mathrm{BR}_i(s_{-i}^*).$$

Example 2.7 (Chicken Game)**.** Two drivers drive towards each other in a one lane road. Each driver has to decide wether to swerve (S) or not (N). If none of them changes the direction, they both collide.

Driver 2

Driver 1	S	N
S	$0, 0$	$-1, 1$
N	$1, -1$	$-100, -100$

Chicken Game

Example 2.8 (Matching pennies)**.** Two individuals have a coin each. They simultaneously choose heads tails and show their coin. If both coins are matched player 1 pays \$1 to player 2. Otherwise, player 2 pays \$1 to player 1.

Player 2

Player 1	H	T
H	$-1, 1$	$1, -1$
T	$1, -1$	$-1, 1$

Matching pennies

No NE in pure strategies.

Some Remarks about NE.

1. A NE is part of the rationalizable strategies.
2. If the IESD procedure yields a unique strategy for each player, then these strategies constitute the unique NE of the game.

3. However a NE may include weakly dominated strategies.

		Player 2	
		H	T
Player 1	H	$0,0$	$0,0$
	T	$0,0$	$1,1$

4. The unique NE of a game may not be Pareto Optimal. (Prisoner's Dilemma).

5. A game may have more than one NE in pure strategies. (Stug Hunt Game).

6. A game may not have any NE in pure strategies. (Matching pennies).

2.3 Nash Equilibrium in mixed strategies.

Definition 2.9 (Mixed Strategies)**.** Let a game $G = (I, S, u)$. Fix a player $i = 1, 2, \ldots, n$. Suppose the set of strategies $S_i = \{s_1^i, \ldots, s_k^i)$ of agent i has k^i elements. A mixed strategy for player i is a probability distribution $\sigma^i = (\alpha_1^i, \ldots, \alpha_{k^i}^i)$ on S_i such that

- $\alpha_1^i, \ldots, \alpha_{k^i}^i \geq 0$.
- $\alpha_1^i + \ldots + \alpha_{k^i}^i = 1$.

Instead of $\sigma^i = (\alpha_1^i, \ldots \alpha_{k^i}^i)$ we will write

$$\sigma^i = \alpha_1^i s_1^i + \cdots + \alpha_{k^i}^i s_k^i$$

We interpret that player i plays the strategy s_l^i with probability α_l^i. We denote by Δ_i the set of mixed strategies for player i.

- Examples.
- Notation.
- The payoffs extend to mixed strategies in a linear way. Payoffs are expected payoffs.

  That is, Let $\left(\sigma^1, \ldots, \sigma^n\right)$ a mixed strategy profile. Then

$$u_i(\sigma) = \sum_{l_1=1}^{k_1} \sum_{l_2=1}^{k_2} \cdots \sum_{l_n=1}^{k_n} \alpha_{l_1}^1 \alpha_{l_2}^2 \cdots \alpha_{l_n}^n u_i \left(s_{l_1}^1, s_{l_2}^2, \ldots, s_{l_n}^n\right)$$

Definition 2.10 (Best response for mixed strategies)**.** Let a game $G = (I, S, u)$. Fix a player $i = 1, 2, \ldots, n$. Suppose he knows that all the other players are playing the mixed strategy σ_{-i}. We say a mixed strategy $\sigma_i \in \Delta_i$ is a best response for player i to σ_{-i} (and we write $\sigma_i \in \mathrm{BR}_i(\sigma_{-i})$ if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\bar{\sigma}_i, \sigma_{-i}), \quad \text{for every other mixed (or pure) strategy } \bar{\sigma}_i \in \Delta_i$$

Definition 2.11 (Mixed strategy Nash Equilibrium)**.** Let a game $G = (I, S, u)$. A strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$, consisting of mixed strategies, is a (mixed strategy) Nash Equilibrium of G if for every player $i = 1, 2, \ldots, n$ we have that

$$\sigma_i^* \in \mathrm{BR}_i(\sigma_{-i}^*).$$

Definition 2.12 (Nash Equilibrium (NE))**.** Let a normal game $G = (I, S, u)$. A Nash Equilibrium of G is either a pure or a mixed strategy Nash Equilibrium of G.

Observation 2.13 (A property of mixed strategy Nash Equilibria). Suppose σ^* is a mixed strategy NE. Then for each player $i \in N$,

1. given σ^*_{-i}, player player i obtains the same payoff, say v_i, with any action to which σ^*_i assigns **strictly positive probability**. That is player i is indifferent between any of the strategies that he uses with **strictly positive probability** in the NE σ^*.

2. given σ^*_{-i}, player i obtains at most the payoff v_i with any action to which σ^*_i assigns **zero probability**.

Observation 2.14 (Existence Nash Equilibria). Every finite normal game has at least one NE.

Example 2.15 (Employee monitoring)**.** An employee can work (W) or shirk (S). His salary is \$100 unless he is caught shirking. In the latter case the salary is \$0. The cost of working for the employee is \$50.

The manager may monitor (M) the employee or not (N). The cost of monitoring is \$10 . The value of the employee's work for the manager is \$200 if the employee works and \$0 if he shirks.

		Manager	
		M	N
Employee	S	$0, -10$	$100, -100$
	W	$50, 90$	$50, 100$

Employee monitoring

Note that there are no NE in pure strategies. Let us look for a NE in mixed strategies of the form

$$\begin{aligned} \sigma_1 &= xS + (1-x)W, \quad 0 \le x \le 1 \\ \sigma_2 &= yM + (1-y)N \quad 0 \le y, \le 1 \end{aligned}$$

The expected utility of the players in this equilibrium are

$$\begin{aligned} u_1(\sigma_1, \sigma_2) &= 0 \times xy + 100x(1-y) + 50y(1-x) + 50(1-x)(1-y) = 50 + x(50 - 100y) \\ u_2(\sigma_1, \sigma_2) &= -10xy - 100x(1-y) + 90y(1-x) + 100(1-x)(1-y) = 100 - 200x + y(100x - 10) \end{aligned}$$

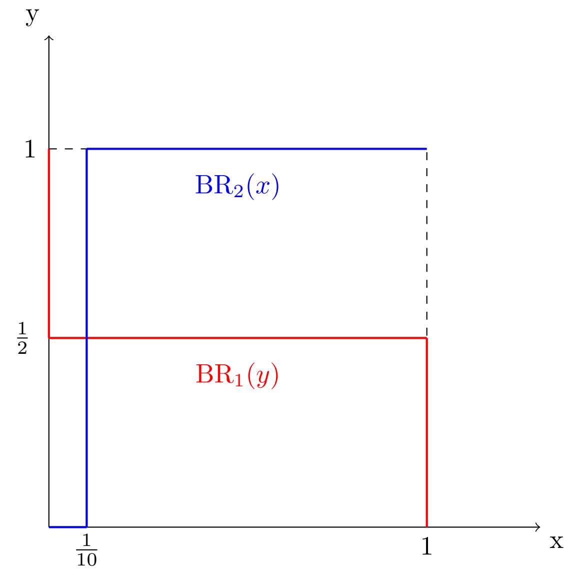
From this, we see that

$$\mathrm{BR}_1(y) = \mathrm{BR}_1(\sigma_2) = \begin{cases} x = 0 & \text{if } y > 1/2 \\ x \in [0,1] & \text{if } y = 1/2 \\ x = 1 & \text{if } y < 1/2 \end{cases}$$

and

$$\mathrm{BR}_2(x) = \mathrm{BR}_2(\sigma_1) = \begin{cases} y = 0 & \text{if } x < 1/10 \\ y \in [0,1] & \text{if } x = 1/10 \\ y = 1 & \text{if } x > 1/10 \end{cases}$$

Graphically,



The intersection of the graphs yields the Nash equilibria of the game. The unique NE is $x^* = \frac{1}{10}$, $y = \frac{1}{2}$. That is,

$$\begin{aligned}\sigma_1^* &= \frac{1}{10}S + \frac{9}{10}W, \\ \sigma_2^* &= \frac{1}{2}M + \frac{1}{2}N\end{aligned}$$

The expected payoffs are $u_1 = \$50$ for the employee and $u_2 = \$80$ for the manager.

Observation 2.16.

1. No NE in pure strategies.
2. A unique NE in mixed strategies. Employee shirks with probability 1/10. Manager monitors with probability 1/2. The expected payoffs are \$50 for the employee and \$80 for the manager.
3. **Both players are indifferent between their strategies. Each player chooses the mixed strategy that makes his opponent indifferent. This yields a general procedure to compute the NE in mixed strategies.**

Example 2.17 (Natural Monopoly)**.** Two firms consider entering a market. The market generates total profits of \$300. There is a fixed cost of entry of \$200. If both firms enter the market they share equally the profifs.

		Firm 2: I	Firm 2: O
Firm 1	I	$-50, -50$	$100, 0$
	O	$0, 100$	$0, 0$

Natural Monopoly

Note that there are two NE in pure strategies: (I, O) and (O, I). Let us look for a NE in mixed strategies of the form

$$\begin{aligned}\sigma_1 &= xI + (1-x)O, \quad 0 \leq x \leq 1 \\ \sigma_2 &= yI + (1-y)O \quad 0 \leq y, \leq 1\end{aligned}$$

The expected utility of the players in this equilibrium are

$$\begin{aligned}u_1(\sigma_1, \sigma_2) &= -50xy + 100x(1-y) + 0 \times y(1-x) + 0 \times (1-x)(1-y) = 100x - 150xy = 10x(10-15y) \\ u_2(\sigma_1, \sigma_2) &= -50xy + 0 \times x(1-y) + 100y(1-x) + 0 \times (1-x)(1-y) = 100y - 150xy = 10y(10-15x)\end{aligned}$$

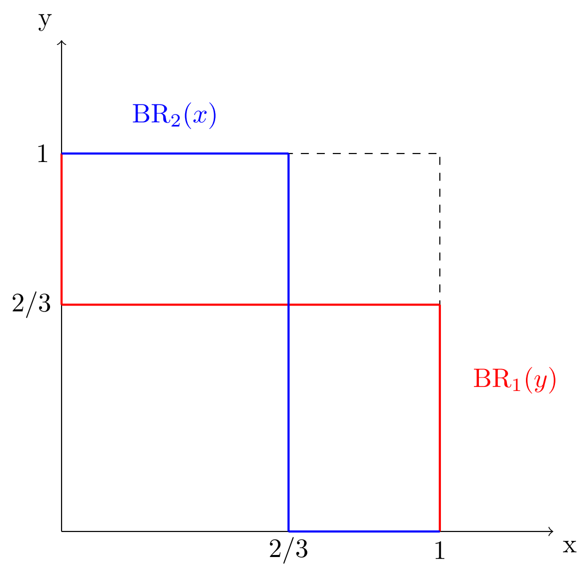
From this, we see that

$$\mathrm{BR}_1(y) = \mathrm{BR}_1(\sigma_2) = \begin{cases} x = 1 & \text{if } y < 2/3 \\ x \in [0,1] & \text{if } y = 2/3 \\ x = 0 & \text{if } y > 2/3 \end{cases}$$

and

$$\mathrm{BR}_2(x) = \mathrm{BR}_2(\sigma_1) = \begin{cases} y = 1 & \text{if } x < 2/3 \\ y \in [0,1] & \text{if } x = 2/3 \\ y = 0 & \text{if } x > 2/3 \end{cases}$$

Graphically,



The intersection of the graphs yields the Nash equilibria of the game. There are three NE,

(a) $x^* = 0$, $y = 1$. That is,

$$\begin{aligned} \sigma_1 &= I \\ \sigma_2 &= O \end{aligned}$$

with payoffs $u_1 = \$100$ $u_2 = \$0$.

(b) $x^* = 1$, $y = 0$. That is,

$$\begin{aligned} \sigma_1 &= O \\ \sigma_2 &= I \end{aligned}$$

with payoffs $u_1 = \$0$ $u_2 = \$100$.

(c) $x^* = y = \frac{2}{3}$. That is,

$$\begin{aligned} \sigma_1 &= \frac{2}{3}I + \frac{1}{3}O, \quad 0 \leq x \leq 1 \\ \sigma_2 &= \frac{2}{3}I + \frac{1}{3}O \quad 0 \leq y, \leq 1 \end{aligned}$$

with payoffs $u_1 = u_2 = \$0$.

Observation 2.18. Note that in the mixed NE we have the following two types of (bad) situations

1. Coordination failure: Both firms enter the market. The probability that this happens is

$$\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$$

2. Loss of opportunity: No firm enters the market. The probability that this happens is

$$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

If both firm coordinate on a coin (for example, firm 1 enters when heads and firm 2 enters when heads) the expected profits are

$$\frac{1}{0} \times 0 + \frac{1}{0} \times 100 = 50$$

3 Bertrand Duopoly with Differentiated Products

We now study the case in which firms offer different products to the market; each firm produces a different good. Firms choose prices.

Suppose there are two firms: firm 1 and firm 2. Each firm chooses the price for its product without knowing the price the other firm has chosen. Prices are denoted by p_1 and p_2, respectively. The cost to firm i of producing quantity q_i is $c_i(q_i) = cq_i$.

Products are differentiated. That is, if f the prevailing prices in the market are (p_1, p_2) the quantity that consumers demand from firm 1 is

$$q_1(p_1, p_2) = \max\{0, a - p_1 + bp_2\}$$

and the quantity that consumers demand from firm 2:

$$q_2(p_1, p_2) = \max\{0, a - p_2 + bp_1\}$$

with $0 \leq b < 2$. Thus, products are different, but they compete in a single market: low prices of the rival's product lowers the demand for my own product.

We may study this situation as a normal game in which players have a continuum of strategies:

1. The set of players is $I = \{\text{Firm } 1, \text{Firm } 2\}$.
2. The sets of strategies are $S_1 = S_2 = [0, +\infty)$.
3. The payoff functions are

$$\begin{aligned} u_1(p_1, p_2) &= q_1(p_1, p_2)(p_1 - c) \\ u_(p_1, p_2) &= q_2(p_1, p_2)(p_2 - c) \end{aligned}$$

   If firms choose prices such that

$$a - p_1 + bp_2 \geq 0, \quad a - p_2 + bp_1 \geq 0$$

   then

$$\begin{aligned} u_1(p_1, p_2) &= (a - p_1 + bp_2)(p_1 - c) \\ u_2(p_1, p_2) &= (a - p_2 + bp_1)(p_2 - c) \end{aligned}$$

   We compute now the NE of the above game. This consists of price pair (p_1^*, p_2^*) such that p_1^* is Firm 1's best response to Firm 2's price p_2^* , and p_2^* is Firm 2's best response to Firm 1's price p_1^*. That is, p_1^* is a solution of

$$\max_{p_1 \geq 0} u_1(p_1, p_2^*) = (a - p_1 + bp_2^*)(p_1 - c)$$

and p_2^* is a solution of

$$\max_{p_2 \geq 0} u_2(p_1^*, p_2) = (a - p_2 + bp_1^*)(p_2 - c)$$

The FOC for firm 1 is

$$\frac{\partial u_1(p_1, p_2^*)}{\partial p_1} = 0$$

That is, $a + c - 2p_1 + bp_2 = 0$ so, we must have

$$p_1^* = \frac{a + c + bp_2^*}{2}$$

Likewise, the FOC for firm 2 yields

$$p_2^* = \frac{a + c + bp_1^*}{2}$$

Note that the SOC hold for both firms

$$\frac{\partial^2 u_1(p_1, p_2^*)}{\partial p_1^2} = \frac{\partial^2 u_2(p_1, p_2^*)}{\partial p_2^2} = -2 < 0$$

Therefore, if (p_1^*, p_2^*) is a NE we must have that

$$\begin{aligned} p_1^* &= \frac{a + c + bp_2^*}{2} \\ p_2^* &= \frac{a + c + bp_1^*}{2} \end{aligned}$$

We obtain

$$p_1^* = p_2^* = \frac{a + c}{2 - b}$$

4 Public goods

Two agent have an initial wealth $w_1 > 1$, $w_2 > 1$. They have to contribute part of their wealth towards a public good. If agent $i = 1, 2$ contributes the amount c_i towards the public good, the utilities of the agents is

$$u_1(c_1, c_2) = \ln(2c_1 + c_2) - c_1, \quad u_2(c_1, c_2) = \ln(c_1 + 2c_2) - c_2$$

We first describe the situation as a static game. The players are $N = \{1, 2\}$. Their strategy sets are $S_1 = [0, w_1]$, $S_2 = [0, w_2]$. And their payoffs are represented by the utility functions above.

We now compute the Nash equilibria of the game. Let us start with the best reply of player 1. We have to compute the solution of the following maximization problem.

$$\max_{c_1} \ln(2c_1 + c_2) - c_1$$

The first order condition is

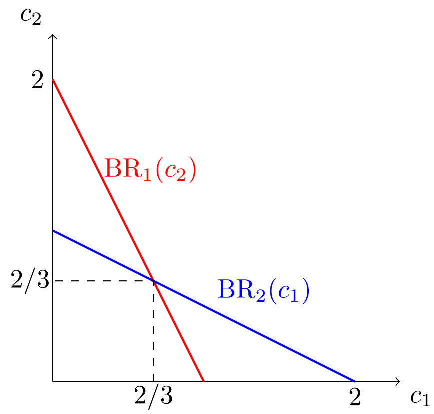
$$\frac{2}{2c_1 + c_2} = 1$$

So,

$$\text{BR}_1(c_2) = \frac{2 - c_2}{2}$$

Similarly the best reply of player 2 is

$$\text{BR}_2(c_2) = \frac{2 - c_1}{2}$$



And the NE have to satisfy the equations

$$\begin{aligned} c_1 &= \frac{2-c_2}{2} \\ c_2 &= \frac{2-c_1}{2} \end{aligned}$$

The solution is

$$c_1^* = c_2^* = \frac{2}{3}$$

The utilities are $u_1^* = u_2^* = \ln(5/3) - 2/3 \approx 0.026$

The social optimal, given by $\max_{c_1,c_2}(u_1(c_1,c_2) + u_2(c_1,c_2))$, is given by the first order conditions

$$\frac{2}{2c_1+c_2} + \frac{1}{c_1+2c_2} = 1$$
$$\frac{1}{2c_1+c_2} + \frac{2}{c_1+2c_2} = 1$$

The solution is $\bar{c}_1 = \bar{c}_2 = 1$. The utilities are $\bar{u}_1 = \bar{u}_2 = \ln(3) - 1 \approx 0.098$. Under the NE the production of the public good is below the optimum.

5 Hotelling/Downs model of electoral competition

Proposed by Hotelling (1929) and Downs (1957).

Voters:

- Each voter v has a favorite policy on $v \in [0,1]$; She has single-peaked preference and her utility decreases as the winner's position is further away from her favorite policy. For example, $u_v(t) = -|v-t|$.

  Each voter will vote sincerely, choosing the party whose position is closest to her favorite policy.

- Voters are distributed according to a distribution function $F(v)$ defined on $[0,1]$.
  - $F(0) = 0$.
  - F is strictly increasing $[0,1]$.
  - $F(1) = 1$.

Political parties:

- There are 2 political parties.
- Parties compete by choosing a policy on the interval $t_1, t_2 \in [0, 1]$.
- The party with most votes wins; if there is a draw, each party has a 50% chance of winning.
- Parties only care about winning, and will commit to the platforms they have chosen.

Median Voter Theorem: Let m be the median voter position, $F(m) = 1/2$. Theres is a unique NE: (m, m).

6 Reporting a crime (From the book of Martin J. Osborne)

A crime is observed by n people. Each person would like the police to be informed, but prefers that somebody else goes through the trouble of informing the police.

We assume that the utility of each agent $i = 1, \ldots, n$ is

- $v > 0$ if somebody else informs the police.
- $v - c > 0$ if agent i informs the police.
- 0 if nobody informs the police.

This is a game with $N = \{1, \ldots, n\}$ agents. The set of strategies of agent $i \in N$ is $S_i = \{I, N\}$. This game has n NE. In each of these NE one of the agents informs the police and no other does. Why?

Let us look for a **symmetric** NE in mixed strategies of the form

$$\sigma = \sigma_i = pI + (1-p)N$$

Then player i is indifferent between I and N. Since,

$$\begin{aligned} u_i(I, \sigma_{-i}) &= v - c \\ u_i(N, \sigma_{-i}) &= 0 \cdot \Pr\{\text{no other agent calls}\} + v \cdot (1 - \Pr\{\text{no other agent calls}\}) \\ &= v \cdot (1 - \Pr\{\text{no other agent calls}\}) \end{aligned}$$

we must have

$$v - c = v \cdot (1 - \Pr\{\text{no other agent calls}\})$$

That is

$$\frac{c}{v} = \Pr\{\text{no other agent calls}\} = (1-p)^{n-1}$$

So,

$$p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$$

Note that $0 < c/v < 1$, so $0 < p < 1$. We conclude that there is a unique symmetric NE in mixed strategies of the form

$$\sigma_i = pI + (1-p)N, \quad p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}, \quad i = 1, \ldots, n$$

Not that,

1. Since,

$$\lim_{n \to \infty} \left(\frac{c}{v}\right)^{\frac{1}{n-1}} = 1$$

   and $\left(\frac{c}{v}\right)^{\frac{1}{n-1}}$ is increasing we have that, as n increases, the probability that a given agent i reports decreases to 0.

2. The probability that in the NE nobody reports to the police is

$$\begin{aligned}
\Pr\{\text{no agent calls}\} &= \Pr\{\text{agent } i \text{ does not call}\} \cdot \Pr\{\text{no other agent calls}\} \\
&= (1-p) \cdot \left(\frac{c}{v}\right)
\end{aligned}$$

Since p decreases to 0 with n, we see that $\Pr\{\text{no agent calls}\}$ increases to $\frac{c}{v}$ with n. That is, the larger the group of witnesses the less likely it is that the police are informed about the crime.